ERN

DAL

YSIS

THE

Reſidual Analyſis;

A

NEW BRANCH

OF THE

ALGEBRAIC ART,

Of very extenſive USE, both in Pure Mathematics, and Natural Philoſophy.

BOOK I.

By JOHN LANDEN.

LONDON,

Printed for the AUTHOR; and ſold by L. HAWES, W. CLARKE, and R. COLLINS, at the *Red Lion* in *Pater-noſter Row.*

MDCCLXIV.

PREFACE.

H*AVING, some time ago, hit upon a new and easy method of investigating the binomial theorem, by a process purely algebraical; I was led to consider whether the means which enabled me to investigate that theorem would not be of use in the investigation of other theorems; and I soon found, that a method of computation depending on such means might be applied in many enquiries. Whereupon, believing such method of computation might be acceptable to the Mathematical World, I, with a view of publishing it, endeavoured to improve it, and to dispose the several articles relating thereto in such order as might conduce to the easy attaining a knowledge thereof. The result of my endeavours is the following treatise; in which the elements and common branches of the analytic art are purposely omitted, upon the supposition that the Reader is previously acquainted therewith; my design being only to teach a particular branch of that art, with its application to Geometry and Natural Philosophy. Which particular branch I have called the* Residual Analysis; *because, in all the enquiries wherein it is made use of, the chief means whereby we obtain the desired conclusions are such quantities, and algebraic expressions, as by Mathematicians are denominated* residuals.

The principles of the common Algebra and Geometry having been thought insufficient to enable the Analyst to pursue his speculations in certain branches of science; new principles, very different from those before made use of, have, through a supposed necessity, been introduced into Analytics. The Fluxionists, following Sir Isaac Newton,

introduce an imaginary motion, and recur to the generation of quantities by a supposed flowing, or continual increase, of their parts. Mr. LEIBNITZ *and his followers, to avoid the supposition of motion, consider quantities as composed of infinitesimal elements; and reject certain parts of the infinitely small increments of quantities as infinitely less than other parts. In the* Residual Analysis, *(admitting no principles but such as were anciently received in Algebra and Geometry,) we neither have recourse to infinitesimals, nor to the principles of motion; but consider magnitudes as already formed, without any regard to their genesis, except in particular cases, where the manner of their being generated may be the proper subject of enquiry: And, as this Analysis is not less (if not more) useful than the fluxionary, or differential, calculus, it will consequently appear, that the Analytic art, founded, and carried on, upon such principles as were anciently received therein, (without the aid of any foreign ones relating to an imaginary motion, or infinitesimals,) is far more extensive than Mathematicians have hitherto reckoned it.*

*Quantities infinitely small, and quantities infinitely less than quantities infinitely little, being incomprehensible; and the rejecting certain quantities as infinitely less than other infinitely little quantities, being, except in approximations, a very unsatisfactory (if not erroneous) way of getting rid of such quantities: The Principles of the method of Infinitesimals are certainly liable to some just objections, which cannot be made to the Principles of Fluxions *; such infinitely small quantities as are almost continually under consideration in the first mentioned method, being no way concerned in the fluxionary doctrine, when it is explained and applied in a proper manner. If any thing can be said by way of objection to the fluxionary method, it is, that the new principles on which it is founded, though accurate, are not the genuine principles of Analytics, for the improvement of which, those principles were borrowed from the doctrine of motion: And that, although such borrowed principles may enable us to give very concise solutions to certain problems, yet perhaps we must not expect to bring the Analytic art to its utmost perfection, otherwise*

* Dr. MATY, comparing the method of Infinitesimals with that of Fluxions, says, " Celle-ci a donc le merite d'une plus grande exactitude, mais cet avantage " est racheté par la necessité qu'elle impose d'avoir recours aux principes du " mouvement." JOURNAL BRITANNIQUE, *Mois de Fevrier* 1751.

than

than by proceeding upon its own proper principles. What weight there may be in ſuch objection, I ſhall not take upon me to determine. Yet I muſt confeſs, that, how natural ſoever it may be, in the reſolution of certain problems relating to geometrical magnitudes, to conſider ſuch magnitudes as generated by motion, it ſeems to me not natural to bring motion into conſideration in reſolving problems purely algebraical: Nor does it ſeem natural, in reſolving problems concerning the motion of bodies, to ſuperinduce imaginary motions, and therewith bring into conſideration the velocity *of* time, *the* velocity *of a* velocity, *&c. nor yet does it appear more natural, in the reſolution of other problems, to make uſe of the fluxionary method, when (as is moſt commonly the caſe in that doctrine) the fluxions introduced into the proceſs can only in a figurative ſenſe be ſaid to be the velocities of increaſe of the quantities called their fluents; ſuch figurative expreſſion being not the natural language of Analytics, but frequently, inſtead of conveying clear and diſtinct ideas, is confuſedly employed in treating of quantities as generated by motion, which in reality cannot be conceived to be ſo generated*. Therefore I am induced to think, that, not only in the reſolution of problems purely algebraical, but likewiſe in Geometry and Natural Philoſophy, when an analytical proceſs is requiſite, and what is called the common algebra is inſufficient, the* Reſidual Analyſis, *which is founded (as I conceive) on the genuine principles of Analytics, is, for the moſt part, more properly applicable than the Fluxionary Analyſis, which is founded on new principles borrowed as above mentioned. But however, very far from being poſitive in this matter, I freely ſubmit it to the Judgment of the Public.*

In comparing the methods of computation juſt now mentioned, it may be obſerved, that, where the direct method of Fluxions might be applied, we, in applying our Analyſis, compute the value of the quotient of one reſidual divided by another; and, where the inverſe method of Fluxions would be applicable, we, from a given reſidual diviſor and a particular value of a certain quotient, aſſign the correſpondent dividend, of a particular Reſidual form. Now, the particular value, which we have occaſion to conſider of ſuch quotient, being always to unity, as the fluxion of the firſt

* Such quantities are *weight*, *denſity*, &c.

member

member of the dividend to the fluxion of the first member of the divisor; it therefore often happens, that, though we proceed upon different principles, part of our process is the same as if we had pursued the fluxionary method: So that, in fact, many of the articles in this treatise may be of the same use in the doctrine of Fluxions as in this Analysis; and probably some of those many articles may conduce to the improvement of that doctrine, if the Reader, having studied it, should chuse to adhere to it.

The SUBSCRIBERS to this WORK, whose Names are at present, come to Hand, are

ROBERT Austin, *Esq;*
Mr. Thomas Allen, *Teacher of the Mathematics, in Spalding,* 6 copies.

Edward Brown, *Esq;*
Charles Balguy, M. D.
Mr. Abraham Beharrell.
Mr. Abraham Baley.
Mr. Stephen Bee, *Steward, and Land-Surveyor.*
Thomas Barker, *Esq;*
The Rev. Stanhope Bruce, A. M.
Mr. Reg. Brathwaite, *of St. John's College, Cambridge.*
The Rev. Mr. Barr, A. M.
Mr. Charles Brent.
Mr. William Brailsford.
Daniel Bayley, *Gent.*
Captain James Barnes.
Mr. William Bevil, 2 copies.
Mr. Anthony Birks, *Master of the Free-School in Donington, Lincolnshire.*
Mr. William Bothamley, *Clockmaker in Spalding.*
Mr. Samuel Beeken.

The Right Hon. the Lord Carysfort, *one of the Lords of the Admiralty.*
The Right Hon. Sir John Cust, *Bart. Speaker of the House of Commons.*
Jonathan Cope, *Esq;*
John Calcraft, *Esq;*
Charles Causton, *Esq;*
Charles Cæsar, *Esq;*
James Collier, *Esq;*
The Rev. J. Colson, A. M. F. R. S. *late Lucasian Professor of Mathematics in the University of Cambridge.*
Robert Clarke, *Gent.*
The Rev. Mr. James Clarke.
Mr. Martin Cole.
Mr. William Caldecott, *Surgeon.*
Mr. William Clifton.
Mr. John Corbet.

Daniel Douglas, *Gent.*
Dominick Donnelly, *Gent.*
Mr. John Dickenson.

Mr. Richard Edwards.
Mr. Langley Edwards.

The Right Hon. the Countess Dowager Fitzwilliam.
Samuel Forster, *of Grantham, Gent.*
Mr. B. Fernor, *Professor of Mathematics, and Member of the Royal Academy of Sciences at Stockholm.*

The Rev. Mr. Gibson, *late Prebendary of Peterborough.*
The Rev. Mr. Kennet Gibson.

William Hanbury, *of Kelmarsh, Esq;*
Mr. William Hutchinson.
Mr. John Hyde.
Mr Henry Hetley, *of St. John's College, Cambridge.*
Mr. Edward Hare.

The Rev. John Image, A. M.
Mr. William Jepson.

Robert Kelham, *Gent.*

The Right Hon. the Earl of Ludlow.
The Rev. Dr. Robert Lamb, *Dean of Peterborough.*
Mr. William Lax.

Thomas Moore, jun. *Gent.*
The Rev. George Moore, A. M.
Mr. William Moore.
The Rev. Mr. Mirehouse.
The Rev. Mr. Murhall, A. M. *Fellow of Christ College, Cambridge.*
Mr. Francis Maseres, A. M.

Mr. Edward Nott, *Bookseller, and Teacher of the Mathematics, in Stamford.*

Walden Orme, *Esq;*

Armstead Parker, *Esq;*
Peterborough Society.
The Rev. Peter Peckard, A. M.
Mr. John Parish.
Mr. John Perry.
Mr. Joseph Priestley.

Mr. Thomas Quanborough.

Mr. Edward Rollinson

Mr. Richard Robinson, *Land-Surveyor.*

Dr. Robert Smith, F. R. S. *Master of Trinity College, Cambridge.*
Robert Smith, LL. B.
James Scawen, *of Maidwell, Esq;*
Edward Southwell, *Esq;*
Major St. Leger, *Fellow of St. Peter's College, Cambridge.*
The Hon. —— Sandys, *Esq;*
Mr. Thomas Simpson, F. R. S. *and Member of the Royal Academy of Sciences at Stockholm.*
Mr. Benjamin Smith.
Mr. Stephens, *of Deal.*
Mr. Smith, *Steward, and Land-Surveyor.*
Mr. Henry Smith, *Attorney at Law in Stamford.*
The Rev. Mr. John Sanderson.
The Rev. Mr. Anthony Sanderson.
Mr. William Smith.
Mr. Robert Sharman.
Mr. Thomas Saunder.

Thomas Truman, *Esq;*
Richard Tryce, *Esq;*
Trinity College, Cambridge, *for the Library.*
Mr. Tyson, C. C. C. C.
The Rev. Mr. Brownlowe Toller, 2 copies.
The Rev. Mr. Richard Turner, *Teacher of Geometry, Astronomy, and Philosophy, at Worcester.*
Mr. John Thistleton.
Mr. Daniel Tenand, *Writing-Master and Accomptant, in Islington.*
Mr. Thomas Taylor, *Writing-Master and Accomptant, in Shadwell.*
Mr. Tape.
Mr. John Thistlewood, *Land-Surveyor.*

Sir Philip Vavazor, *Knt.*

The Right Rev. Dr. John Thomas, *Lord Bishop of Winchester.*
Lady Charlotte Wentworth.
Matthew Wyldbore, *Esq;*
Thomas Whichcot, *Esq; Member of Parliament for the County of Lincoln.*
The Rev. Mr. Weston, *Rector of Thirfield,* &c.
The Rev. Mr. Waring, *Lucasian Professor of Mathematics in the University of Cambridge.*
Mr. Charlton Wyldbore.
Mr. Thomas White.
Mr. Benjamin Webb, *Writing-Master and Accomptant, in Bunhill-Row.*
Mr. Richard Willson.
Captain Henry Watson.
Captain Henry Wilson.
Mr. Edmund Webster.
Mr. John Wing.
Mr. Tycho Wing, *Mathematical-Instrument-Maker, in London.*
Mr. J. Watts, *of Comb-Abbey,* 2 copies.
Mr. Henry Woolley, *Master of the Academy in Northampton.*
Mr. Richard Weston, *Teacher of the Mathematics, in Yaxley.*
Mr. Jekyl Wilson.
Mr. Graham Wilkinson.
Mr. Joseph Wilkinson.

☞ Purchasers of this First Book will be esteemed Subscribers, and their Names (if they please to give them to the Author, or Publishers) shall be inserted in a complete List, to be printed with the Second Book: —In which Book, the Usefulness of this Analysis will be exemplified, in a variety of very curious Mechanical, and Physico-Geometrical, Enquiries..

J. LANDEN.

Milton, near *Peterborough,*
March 25, 1764.

THE

RESIDUAL ANALYSIS.

CHAP. I.

Terms *and* Characters *explained.*

IT will be proper, in the firſt place, to give an explanation of certain *Terms* and *Characters*, which, in the courſe of this work, we ſhall have frequent occaſion to make uſe of:—accordingly, ſuch explanation is here premiſed.

I.

In any proceſs, a quantity that is conſidered as always retaining the ſame value, is called a *determinate* or *invariable* quantity. And a quantity that is not conſider'd as always retaining the ſame value, but may be taken of any value whatever, or of any value between certain limits, (ſome other quantity, or quantities, concerned therewith, at the ſame time, remaining invariable,) is called an *indeterminate* or *variable* quantity.

2.

An algebraic expreſſion compoſed, in any manner, of any power or powers of any variable quantity, with any invariable coefficients, is called a *function* of that quantity.

For inſtance, $a+bx^m+cx^n$ and $\frac{ex^r+\sqrt{f^2+x^2}}{gx^s}$ are functions of the variable quantity x: And, y being equal to any function of x, $\frac{axy+y\sqrt{x^2+y^2}}{x+b\sqrt{c+y}}$ may be alſo conſidered as a function of x.

If y *be equal to any function of* x, *it is obvious that* x *will be equal to ſome function of* y. *And if* y *and* z *be equal to any functions of* x, x *and* y *may each be conſidered as a function of* z; *and* x *and* z, *each as a function of* y.

3.

If, in any given expreſſion or function of x, wherein $x_{\prime}$ is not concerned, $x_{\prime}$ be ſubſtituted inſtead of x, the given expreſſion and that which reſults from ſuch ſubſtitution are called *ſimilar functions* of x and $x_{\prime}$ reſpectively.

For inſtance, $\frac{ex^p+\sqrt{f^2+x^2}}{gx^r}$ and $\frac{ex_{\prime}^p+\sqrt{f^2+x_{\prime}^2}}{gx_{\prime}^r}$ are ſimilar functions of x and $x_{\prime}$ reſpectively; the values of e, f, g, p, and r being independent of the values of x and $x_{\prime}$: And, y and $y_{\prime}$ being ſimilar functions of x and $x_{\prime}$ reſpectively, $ax^2y+by^m\sqrt{c^2+xy^n}$ may be conſidered as a function of x, and $ax_{\prime}^2y_{\prime}+by_{\prime}^m\sqrt{c^2+x_{\prime}y_{\prime}^n}$ as a ſimilar function of $x_{\prime}$; or the ſame expreſſions ($ax^2y+by^m\sqrt{c^2+xy^n}$ and $ax_{\prime}^2y_{\prime}+by_{\prime}^m\sqrt{c^2+x_{\prime}y_{\prime}^n}$) may be conſidered as ſimilar functions of y and $y_{\prime}$ reſpectively;

a,

a, b, c, m, and n being determinate, whilſt x and $x_{,}$ are indeterminate.

4.

y being any function of x, and $y_{,}$ a ſimilar function of $x_{,}$; the value of the quotient of $y - y_{,}$ divided by $x - x_{,}$, in the particular caſe when $x_{,}$ is equal to x, is called the *ſpecial value* of that quotient; and x and y are reſpectively named the *prime member of the diviſor*, and the *prime member of the dividend.*

The ſaid quotient, which algebraiſts commonly denote by $\frac{y - y_{,}}{x - x_{,}}$ or $\overline{y - y_{,}} \div \overline{x - x_{,}}$, we ſhall, for brevity ſake, ſometimes denote by $[x \mid y]$; and the ſpecial value thereof we ſhall expreſs by $[x \perp y]$.

Moreover, $[x \perp y]$ and $[x_{,} \perp y_{,}]$ being ſimilar functions of x and $x_{,}$, the ſpecial value of $\overline{[x \perp y] - [x_{,} \perp y_{,}]} \div \overline{x - x_{,}}$ will be denoted by $[x \perp\!\!\perp y]$:

And, in like manner, $[x \perp\!\!\perp y]$ and $[x_{,} \perp\!\!\perp y_{,}]$ being ſimilar functions of x and $x_{,}$, the ſpecial value of $\overline{[x \perp\!\!\perp y] - [x_{,} \perp\!\!\perp y_{,}]} \div \overline{x - x_{,}}$ will be denoted by $[x \perp\!\!\perp\!\!\perp y]$: &c.

And the like is to be underſtood, when, for any quantities having the like relation, other letters or characters are ſubſtituted inſtead of x, $x_{,}$, y, and $y_{,}$.

Accordingly $[x_{,} \perp y_{,}]$ will denote the ſpecial value of $\overline{y_{,} - y_{,,}} \div \overline{x_{,} - x_{,,}}$, $y_{,}$ and $y_{,,}$ being ſimilar functions of $x_{,}$ and $x_{,,}$ reſpectively: And, $[x_{,} \perp y_{,}]$ and $[x_{,,} \perp y_{,,}]$ being alſo ſimilar functions of $x_{,}$

 and

and $x_{\prime\prime}$, the ſpecial value of $\overline{[x_{\prime} \perp y_{\prime}] - [x_{\prime\prime} \perp y_{\prime\prime}]} \div \overline{x_{\prime} - x_{\prime\prime}}$ will be expreſſed by $[x_{\prime} \perp\!\!\perp y_{\prime}]$: &c.

So $[y \perp z]$ will ſignify the ſpecial value of $\overline{z - z_{\prime}} \div \overline{y - y_{\prime}}$, z and $z_{\prime}$ being ſimilar functions of y and $y_{\prime}$ reſpectively:

And $[y \perp z]$ and $[y_{\prime} \perp z_{\prime}]$ being alſo ſimilar functions of y and $y_{\prime}$, $[y \perp\!\!\perp z]$ will expreſs the ſpecial value of $\overline{[y \perp z] - [y_{\prime} \perp z_{\prime}]} \div \overline{y - y_{\prime}}$: &c.

5.

Let it be noted, that, whenever y denotes any function of x; $y_{\prime}$, $y_{\prime\prime}$, $y_{\prime\prime\prime}$, &c. ſhall denote ſimilar functions of x, $x_{\prime}$, $x_{\prime\prime}$, $x_{\prime\prime\prime}$, &c. reſpectively.

Likewiſe, whenever z denotes any function of y; z, $z_{\prime}$, $z_{\prime\prime}$, $z_{\prime\prime\prime}$, &c. ſhall denote ſimilar functions of y, $y_{\prime}$, $y_{\prime\prime}$, $y_{\prime\prime\prime}$, &c. reſpectively. And the like is to be obſerved, with reſpect to other quantities.

6.

When any quantities v, x, y, z, &c. are concerned in any proceſs, and the quantities $v_{\prime}$, $x_{\prime}$, $y_{\prime}$, &c. or $[v \perp x]$, $[v \perp y]$, &c. are alſo concerned in the ſame proceſs; it muſt be underſtood, that x, y, &c. are each conſider'd as ſome function of v, though their being ſo related be not there particularly mentioned.

THE

RESIDUAL ANALYSIS.

CHAP. II.

Of the Invention of RULES *neceſſary to faciliate Computations in this Analyſis.*

IN making computations by the method taught in this Treatiſe, we ſhall frequently have occaſion to aſſign the quotient of $F - F_{,}$ divided by $x - x_{,}$, F being ſome function of x; or, at leaſt, we ſhall often find it neceſſary to have recourſe to ſome Rule derived from a conſideration of the relation between the ſpecial value of ſuch quotient and the prime member of the correſpondent dividend. Therefore, as the Books of Algebra hitherto publiſhed are deficient in reſpect to ſuch diviſion, it will here be proper to ſhew how ſuch quotient may always be obtained; and to deduce therefrom ſuch Rules as may be uſeful to facilitate computations in the ſucceeding Chapters.

I.

The Theorems which chiefly enable us to perform the diviſion above mentioned are the following, viz.

$$\frac{v^{\frac{m}{r}} - w^{\frac{m}{r}}}{v - w}$$

$$\frac{v^{\frac{m}{r}} - w^{\frac{m}{r}}}{v - w} = \frac{v^{m-1} + v^{m-2}w + v^{m-3}w^2 + v^{m-4}w^3 \ (m)}{v^{m-\frac{m}{r}} + v^{m-\frac{2m}{r}}w^{\frac{m}{r}} + v^{m-\frac{3m}{r}}w^{\frac{2m}{r}} + v^{m-\frac{4m}{r}}w^{\frac{3m}{r}} \ (r)} \ *$$

$$= v^{\frac{m}{r}-1} \times \frac{1 + \frac{w}{v} + \left.\overline{\frac{w}{v}}\right|^{2} + \left.\overline{\frac{w}{v}}\right|^{3} \ (m)}{1 + \left.\overline{\frac{w}{v}}\right|^{\frac{m}{r}} + \left.\overline{\frac{w}{v}}\right|^{\frac{2m}{r}} + \left.\overline{\frac{w}{v}}\right|^{\frac{3m}{r}} \ (r)},$$

m and r being positive integers;

and $uw - \underset{,}{u}\underset{,}{w} = w \times \overline{u - \underset{,}{u}} + \underset{,}{u} \times \overline{w - \underset{,}{w}}$;

which, by an easy multiplication, will be found to be true.

From the first equation it follows, that $\frac{v^{-\frac{m}{r}} - w^{-\frac{m}{r}}}{v - w} = -$

$$\frac{v^{\frac{m}{r}} - w^{\frac{m}{r}}}{\left.\overline{vw}\right|^{\frac{m}{r}} \times \overline{v - w}} \text{ is } = -v^{-1}w^{-\frac{m}{r}} \times \frac{1 + \frac{w}{v} + \left.\overline{\frac{w}{v}}\right|^{2} + \left.\overline{\frac{w}{v}}\right|^{3} \ (m)}{1 + \left.\overline{\frac{w}{v}}\right|^{\frac{m}{r}} + \left.\overline{\frac{w}{v}}\right|^{\frac{2m}{r}} + \left.\overline{\frac{w}{v}}\right|^{\frac{3m}{r}} \ (r)}.$$

---

* This theorem may be investigated as follows.
It is well known, that

$$\frac{v^{m} - w^{m}}{v - w} \text{ is } = v^{m-1} + v^{m-2}w + v^{m-3}w^2 \ (m),$$

$$\text{and } \frac{a^{r} - b^{r}}{a - b} = a^{r-1} + a^{r-2}b + a^{r-3}b^2 \ (r),$$

m and r being positive Integers.

In the second equation write $v^{\frac{m}{r}}$ and $w^{\frac{m}{r}}$ instead of a and b respectively, and you will have

$$\frac{v^{m} - w^{m}}{v^{\frac{m}{r}} - w^{\frac{m}{r}}} = v^{m-\frac{m}{r}} + v^{m-\frac{2m}{r}}w^{\frac{m}{r}} + v^{m-\frac{3m}{r}}w^{\frac{2m}{r}} \ (r).$$

Then, from the first and third equations, it will appear by division, that

$$\frac{v^{\frac{m}{r}} - w^{\frac{m}{r}}}{v - w} \text{ is } = \frac{v^{m-1} + v^{m-2}w \ (m)}{v^{m-\frac{m}{r}} + v^{m-\frac{2m}{r}}w^{\frac{m}{r}} \ (r)}:$$

Which being so obvious, it is matter of surprize to me, that Algebraists have not before observed it, and shewn its singular use in Analytics.

EXAMPLE

EXAMPLE I. Taking $\frac{m}{r} = \frac{4}{3}$ we have

$$\overline{v^{\frac{4}{3}} - w^{\frac{4}{3}}} \div \overline{v-w} = v^{\frac{1}{3}} \times \frac{1 + \frac{w}{v} + \left.\overline{\frac{w}{v}}\right|^{2} + \left.\overline{\frac{w}{v}}\right|^{3}}{1 + \left.\overline{\frac{w}{v}}\right|^{\frac{4}{3}} + \left.\overline{\frac{w}{v}}\right|^{\frac{8}{3}}}.$$

Whence it is evident, that, when w is $= v$, the quotient of $v^{\frac{4}{3}} - w^{\frac{3}{4}}$ divided by $v - w$ is equal to $\frac{4}{3}v^{\frac{1}{3}}$.

Moreover, $\frac{4}{3}$ being $= 1.3333$ &c.

$$\overline{v^{\frac{4}{3}} - w^{\frac{4}{3}}} \div \overline{v-w} \text{ is} = v^{\frac{1}{3}} \times \frac{1 + \frac{w}{v} + \left.\overline{\frac{w}{v}}\right|^{2} + \left.\overline{\frac{w}{v}}\right|^{3} \ (13333)}{1 + \left.\overline{\frac{w}{v}}\right|^{\frac{4}{3}} + \left.\overline{\frac{w}{v}}\right|^{\frac{8}{3}} + \left.\overline{\frac{w}{v}}\right|^{\frac{12}{3}} \ (10000)} \text{ nearly;}$$

$$\text{or, more nearly, } = v^{\frac{1}{3}} \times \frac{1 + \frac{w}{v} + \left.\overline{\frac{w}{v}}\right|^{2} + \left.\overline{\frac{w}{v}}\right|^{3} \ (133333)}{1 + \left.\overline{\frac{w}{v}}\right|^{\frac{4}{3}} + \left.\overline{\frac{w}{v}}\right|^{\frac{4}{3}} + \left.\overline{\frac{w}{v}}\right|^{\frac{12}{3}} \ (100000)};$$

&c.

Hence it is likewise evident, that, when w is $= v$, the quotient of $v^{\frac{4}{3}} - w^{\frac{4}{3}}$ divided by $v - w$ is equal to $\frac{4}{3}v^{\frac{1}{3}}$: For, the ratio of 1333 &c. to 1000 &c. being as 1.333 &c. to 1, the ultimate value of $\frac{1+1+1+1 \ (1333 \ \&c.)}{1+1+1+1 \ (1000 \ \&c.)}$ (the value of $\frac{1 + \frac{w}{v} + \left.\overline{\frac{w}{v}}\right|^{2} \ (1333 \ \&c.)}{1 + \left.\overline{\frac{w}{v}}\right|^{\frac{4}{3}} + \left.\overline{\frac{w}{v}}\right|^{\frac{8}{3}} \ (1000 \ \&c.)}$ when w is equal to v) is manifestly equal to $\frac{4}{3}$, the quantity from which (by division) 1.333 &c. is obtained.

EXAMPLE

Example II. Taking $\frac{m}{r} = \sqrt{2} = 1.4142$ &c. we have

$$\overline{v^{\sqrt{2}} - w^{\sqrt{2}}} \div \overline{v - w} = v^{\sqrt{2}-1} \times \frac{1 + \frac{w}{v} \quad + \left.\overline{\tfrac{w}{v}}\right|^{2} \quad + \left.\overline{\tfrac{w}{v}}\right|^{3} \quad (1414)}{1 + \left.\overline{\tfrac{w}{v}}\right|^{\sqrt{2}} + \left.\overline{\tfrac{w}{v}}\right|^{2\sqrt{2}} + \left.\overline{\tfrac{w}{v}}\right|^{3\sqrt{2}} \quad (1000)} \text{ nearly;}$$

or, more nearly, $= v^{\sqrt{2}-1} \times \dfrac{1 + \frac{w}{v} \quad + \left.\overline{\tfrac{w}{v}}\right|^{2} \quad + \left.\overline{\tfrac{w}{v}}\right|^{3} \quad (14142)}{1 + \left.\overline{\tfrac{w}{v}}\right|^{\sqrt{2}} + \left.\overline{\tfrac{w}{v}}\right|^{2\sqrt{2}} + \left.\overline{\tfrac{w}{v}}\right|^{3\sqrt{2}} \quad (10000)}$;

&c.

Whence it is evident, that, when w is $= v$, the quotient of $v^{\sqrt{2}} - w^{\sqrt{2}}$ divided by $v - w$ is equal to $\sqrt{2} \times v^{\sqrt{2}-1}$: For the ratio of 14142 &c. to 10000 &c. being as 1.4142 &c. to 1, it is manifest, that the ultimate value of $\frac{1+1+1+1 \ (14142 \text{ \&c.})}{1+1+1+1 \ (10000 \text{ \&c.})}$ (the value of $\dfrac{1 + \frac{w}{v} \quad + \left.\overline{\tfrac{w}{v}}\right|^{2} \quad + \left.\overline{\tfrac{w}{v}}\right|^{3} \quad (14142 \text{ \&c.})}{1 + \left.\overline{\tfrac{w}{v}}\right|^{\sqrt{2}} + \left.\overline{\tfrac{w}{v}}\right|^{2\sqrt{2}} + \left.\overline{\tfrac{w}{v}}\right|^{3\sqrt{2}} \quad (10000 \text{ \&c.})}$ when w is equal to v) is equal to $\sqrt{2}$, the quantity from which (by extracting the root) 1.4142 &c. is derived.

Corollary I. Seeing $\dfrac{1 + \frac{w}{v} \quad + \left.\overline{\tfrac{w}{v}}\right|^{2} \quad + \left.\overline{\tfrac{w}{v}}\right|^{3} \quad (m)}{1 + \left.\overline{\tfrac{w}{v}}\right|^{\frac{m}{r}} + \left.\overline{\tfrac{w}{v}}\right|^{\frac{2m}{r}} + \left.\overline{\tfrac{w}{v}}\right|^{\frac{3m}{r}} \quad (r)}$ is equal to $\frac{m}{r}$, when w is $= v$; it is evident, that the special value of $\overline{v^{\frac{m}{r}} - w^{\frac{m}{r}}} \div \overline{v - w}$ is equal to $\frac{m}{r} v^{\frac{m}{r} - 1}$, whether $\frac{m}{r}$ be positive or negative.

Hence, by substituting p, x, and $x_{,}$ instead of $\frac{m}{r}$, v, and w respectively; it appears, that, p being either positive or negative, $[x \,\dot{-}\, x^{p}]$ (the special value of $\overline{x^{p} - x_{,}^{p}} \div \overline{x - x_{,}}$) is $= p x^{p-1}$.

Corollary

COROLLARY II. $\overline{k+av^{\frac{m}{r}}+bv^{\frac{n}{s}}\ \&c.} - \overline{k+w^{\frac{m}{r}}+bw^{\frac{n}{s}}\ \&c.}$ being $= a \times \overline{v^{\frac{m}{r}} - w^{\frac{m}{r}}} + b \times \overline{v^{\frac{n}{s}} - w^{\frac{n}{s}}}$ &c.

$$\overline{\overline{av^{\frac{m}{r}} + bv^{\frac{n}{s}}\ \&c.} - \overline{aw^{\frac{m}{r}} + bw^{\frac{n}{s}}\ \&c.}} \div \overline{v - w}$$

$$\text{is} = av^{\frac{m}{r}-1} \times M + bv^{\frac{n}{s}-1} \times N\ \&c.$$

$$\text{and } \overline{\overline{av^{-\frac{m}{r}} + bv^{-\frac{n}{s}}\ \&c.} - \overline{aw^{-\frac{m}{r}} + bw^{-\frac{n}{s}}\ \&c.}} \div \overline{v - w}$$

$$= -av^{-1}w^{-\frac{m}{r}} \times M - bv^{-1}w^{-\frac{n}{s}} \times N\ \&c.$$

$$M \text{ being} = \frac{1 + \frac{w}{v} + \overline{\frac{w}{v}}\Big|^{2} + \overline{\frac{w}{v}}\Big|^{3} \quad (m)}{1 + \overline{\frac{w}{v}}\Big|^{\frac{m}{r}} + \overline{\frac{w}{v}}\Big|^{\frac{2m}{r}} + \overline{\frac{w}{v}}\Big|^{\frac{3m}{r}} \quad (r)},$$

$$\text{and } N = \frac{1 + \frac{w}{v} + \overline{\frac{w}{v}}\Big|^{2} + \overline{\frac{w}{v}}\Big|^{3} \quad (n)}{1 + \overline{\frac{w}{v}}\Big|^{\frac{n}{s}} + \overline{\frac{w}{v}}\Big|^{\frac{2n}{s}} + \overline{\frac{w}{v}}\Big|^{\frac{3n}{s}} \quad (s)},$$

&c. *&c.*

Moreover,

$[x \perp \overline{k + ax^{p} + bx^{q}\ \&c.}]$ is, in general, $= pax^{p-1} + qbx^{q-1}$ *&c.*

2.

Suppose $\overline{z^{\frac{m}{r}} - z_{,}^{\frac{m}{r}}} \div \overline{z - z_{,}} = F$, and $\overline{z - z_{,}} \div \overline{x - x_{,}} = f$:

Then, $z - z_{,}$ being $= fx - fx_{,}$, $\overline{z^{\frac{m}{r}} - z_{,}^{\frac{m}{r}}} \div \overline{fx - fx_{,}}$ will be $= F$, and $\overline{z^{\frac{m}{r}} - z_{,}^{\frac{m}{r}}} \div \overline{x - x_{,}} = Ff$.

EXAMPLE. Let z be supposed $= ax + x^{\frac{2}{3}}$, and $z_{,} = ax_{,} + x_{,}^{\frac{2}{3}}$:

then, F being (by what is ſaid above) $= z^{\frac{m}{r}-1} \times \dfrac{1 + \frac{\underset{,}{z}}{z} + \left(\frac{\underset{,}{z}}{z}\right)^{2} \quad (m)}{1 + \left(\frac{\underset{,}{z}}{z}\right)^{\frac{m}{r}} + \left(\frac{\underset{,}{z}}{z}\right)^{\frac{2m}{r}} \quad (r)}$,

and $f = a + x^{-\frac{1}{3}} \times \dfrac{1 + \frac{\underset{,}{x}}{x}}{1 + \left(\frac{\underset{,}{x}}{x}\right)^{\frac{2}{3}} + \left(\frac{\underset{,}{x}}{x}\right)^{\frac{4}{3}}}$, $\overline{z^{\frac{m}{r}} - \underset{,}{z}^{\frac{m}{r}}} \div \overline{x - \underset{,}{x}}$

will be

$$= z^{\frac{m}{r}-1} \times \frac{1 + \frac{\underset{,}{z}}{z} + \left(\frac{\underset{,}{z}}{z}\right)^{2} \quad (m)}{1 + \left(\frac{\underset{,}{z}}{z}\right)^{\frac{m}{r}} + \left(\frac{\underset{,}{z}}{z}\right)^{\frac{2m}{r}} \quad (r)} \times \overline{a + x^{-\frac{1}{3}} \times \frac{1 + \frac{\underset{,}{x}}{x}}{1 + \left(\frac{\underset{,}{x}}{x}\right)^{\frac{2}{3}} + \left(\frac{\underset{,}{x}}{x}\right)^{\frac{4}{3}}}}.$$

Corollary. When $\underset{,}{x}$ is $= x$, $\underset{,}{z}$ will be $= z$, and $f = [x \perp z]$, z being any function of x; moreover, by Cor. 1. of the preceding Article, F will be $= pz^{p-1}$, p being written inſtead of $\frac{m}{r}$.

Therefore $[x \perp z^{p}]$ will be $= pz^{p-1} \times [x \perp z]$.

Example I. Suppoſe $z = a + bx^{m} + cx^{n}$ &c. Then, $[x \perp z]$, by the preceding Article, being $= bmx^{m-1} + cnx^{n-1}$ &c. $[x \perp \overline{a + bx^{m} + cx^{n} \text{ \&c.}}|^{p}]$ will be $=$ $p \times \overline{a + bx^{m} + cx^{n} \text{ \&c.}}|^{p-1} \times \overline{bmx^{m-1} + cnx^{n-1} \text{ \&c.}}$

Suppoſing the coefficients c, d, &c. each $= 0$, we have

$$[x \perp \overline{a + bx^{m}}|^{p}] = bmp \times \overline{a + bx^{m}}|^{p-1} \times x^{m-1}.$$

Example

EXAMPLE II. Let z be supposed $= cx^n + \overline{a+bx^m}|^p$. Then $\overline{cx^n + \overline{a+bx^m}|^p} - \overline{cx_\prime^n + \overline{a+bx_\prime^m}|^p}$ being $= c.\overline{x^n - x_\prime^n} + \overline{\overline{a+bx^m}|^p - \overline{a+bx_\prime^m}|^p}$, $[x \perp \overline{cx^n + \overline{a+bx^m}|^p}]$ will be $= [x \perp cx^n] + [x \perp \overline{a+bx^m}|^p] = cnx^{n-1} + bmp \times \overline{a+bx^m}|^{p-1} \times x^{m-1}$, and $[x \perp \overline{cx^n + \overline{a+bx^m}|^p}|^q] = q \times \overline{cx^n + \overline{a+bx^m}|^p}|^{q-1} \times \overline{cnx^{n-1} + bmp \times \overline{a+bx^m}|^{p-1} \times x^{m-1}}$.

3.

Seeing $uw - u_\prime w_\prime$ is $= w \times \overline{u - u_\prime} + u_\prime \times \overline{w - w_\prime}$, as we have before observed; it must follow, that $\overline{uw - u_\prime w_\prime} \div \overline{v - v_\prime}$ is $= w \times \frac{u - u_\prime}{v - v_\prime} + u_\prime \times \frac{w - w_\prime}{v - v_\prime}$.

EXAMPLE. By taking u, w, $u_\prime$, and $w_\prime$ respectively equal to $\overline{a^2 - v^2}|^{\frac{1}{2}}$, $a^{\frac{1}{2}} + v^{\frac{1}{2}}$, $\overline{a^2 - v_\prime^2}|^{\frac{1}{2}}$, and $a^{\frac{1}{2}} + v_\prime^{\frac{1}{2}}$, we have $\overline{\overline{a^2 - v^2}|^{\frac{1}{2}} \times \overline{a^{\frac{1}{2}} + v^{\frac{1}{2}}} - \overline{a^2 - v_\prime^2}|^{\frac{1}{2}} \times \overline{a^{\frac{1}{2}} + v_\prime^{\frac{1}{2}}}} \div \overline{v - v_\prime} =$
$- \overline{a^{\frac{1}{2}} + v^{\frac{1}{2}}} \times \frac{v + v_\prime}{\sqrt{a^2 - v^2} + \sqrt{a^2 - v_\prime^2}} + \sqrt{a^2 - v_\prime^2} \times \frac{1}{v^{\frac{1}{2}} + v_\prime^{\frac{1}{2}}}$;
$\overline{\overline{a^2 - v^2}|^{\frac{1}{2}} - \overline{a^2 - v_\prime^2}|^{\frac{1}{2}}} \div \overline{v - v_\prime}$ (by what is said above) being equal to $- \frac{v + v_\prime}{\sqrt{a^2 - v^2} + \sqrt{a^2 - v_\prime^2}}$, and $\overline{a^{\frac{1}{2}} + v^{\frac{1}{2}} - \overline{a^{\frac{1}{2}} + v_\prime^{\frac{1}{2}}}} \div \overline{v - v_\prime}$ $(= \overline{v^{\frac{1}{2}} - v_\prime^{\frac{1}{2}}} \div \overline{v - v_\prime})$ equal to $\frac{1}{v^{\frac{1}{2}} + v_\prime^{\frac{1}{2}}}$.

Corollary. When $v_{,}$ is $= v$, $u_{,}$ and $w_{,}$ will be equal to u and w respectively, these last quantities being any functions of v. It is manifest therefore, that

$$[v \perp uw] \text{ will be } = w\,[v \perp u] + u\,[v \perp w].$$

Example I. Supposing u and w to be respectively equal to x^m and y^n, (*i. e.* supposing x and y to be equal to any functions of v,) we have $[v \perp x^m y^n] = y^n \times [v \perp x^m] + x^m \times [v \perp y^n]$ $= mx^{m-1}y^n\,[v \perp x] + nx^m y^{n-1}\,[v \perp y]$.

Example II. Supposing v, u, and w equal to x, cx^m, and $\overline{a+bx^r}\big|^{n}$ respectively, we have

$$[x \perp cx^m \times \overline{a+bx^r}\big|^{n}] = \overline{a+bx^r}\big|^{n} \times [x \perp cx^m] + cx^m \times [x \perp \overline{a+bx^r}\big|^{n}]$$
$$= cm \times \overline{a+bx^r}\big|^{n} \times x^{m-1} + cnx^m \times \overline{a+bx^r}\big|^{n-1} \times [x \perp a+bx^r]$$
$$= cm \times \overline{a+bx^r}\big|^{n} \times x^{m-1} + bcnr \times \overline{a+bx^r}\big|^{n-1} \times x^{m+r-1}.$$

Example III. Taking u and w respectively equal to $\overline{a+bx^n}\big|^{m}$ and $\overline{c+dx^r}\big|^{-1}$, we find

$$[v \perp \overline{a+bx^n}\big|^{m} \times \overline{c+dx^r}\big|^{-1}] = \overline{c+dx^r}\big|^{-1} \times [v \perp \overline{a+bx^n}\big|^{m}]$$
$$+ \overline{a+bx^n}\big|^{m} \times [v \perp \overline{c+dx^r}\big|^{-1}] = m \times \overline{c+dx^r}\big|^{-1} \times \overline{a+bx^n}\big|^{m-1}$$
$$\times [v \perp a+bx^n] - \overline{a+bx^n}\big|^{m} \times \overline{c+dx^r}\big|^{-2} \times [v \perp c+dx^r] =$$
$$bmn \times \overline{c+dx^r}\big|^{-1} \times \overline{a+bx^n}\big|^{m-1} \times x^{n-1}[v \perp x] - dr \times \overline{a+bx^n}\big|^{m} \times$$
$\overline{c+dx^r}\big|^{-2} \times x^{r-1}[v \perp x]$; a, b, c, d, m, n, and r being determinate, whilst x is indeterminate.

Example IV. y being any function of x, we have

$$[x \perp \overline{a+bx^n}\big|^{m} \times y^r] = y^r \times [x \perp \overline{a+bx^n}\big|^{m}] + \overline{a+bx^n}\big|^{m} \times [x \perp y^r]$$
$$= my^r \times \overline{a+bx^n}\big|^{m-1} \times [x \perp a+bx^n] + ry^{r-1} \times \overline{a+bx^n}\big|^{m} \times [x \perp y]$$
$$= bmnx^{n-1}y^r \times \overline{a+bx^n}\big|^{m-1} + ry^{r-1} \times \overline{a+bx^n}\big|^{m} \times [x \perp y].$$

Example

EXAMPLE V.

$$[x \overset{\prime}{-} z\,[x \overset{\prime}{-} z]] \text{ is } = [x \overset{\prime}{-} z]^2 + z\,[x \overset{\prime\prime}{-} z],$$
$$[x \overset{\prime}{-} z\,[x \overset{\prime\prime}{-} z]] = [x \overset{\prime}{-} z] \times [x \overset{\prime\prime}{-} z] + z\,[x \overset{\prime\prime\prime}{-} z],$$
$$[x \overset{\prime}{-} y\,[x \overset{\prime}{-} z]] = [x \overset{\prime}{-} y] \times [x \overset{\prime}{-} z] + y\,[x \overset{\prime\prime}{-} z], \text{ and}$$
$$[x \overset{\prime}{-} [x \overset{\prime}{-} y] \times [x \overset{\prime}{-} z]] = [x \overset{\prime\prime}{-} y] \times [x \overset{\prime}{-} z] + [x \overset{\prime}{-} y] \times [x \overset{\prime\prime}{-} z].$$

y and z being any functions of x.

4.

$uw - \underset{\prime}{u}\underset{\prime}{w}$ being (as is obferved in the laft Article) $= w \times \overline{u - \underset{\prime}{u}} + \underset{\prime}{u} \times \overline{w - \underset{\prime}{w}} = A$; $uwx - \underset{\prime}{u}\underset{\prime}{w}\underset{\prime}{x}$ will be $= xA + \underset{\prime}{u}\underset{\prime}{w} \times \overline{x - \underset{\prime}{x}} = wx \times \overline{u - \underset{\prime}{u}} + x\underset{\prime}{u} \times \overline{w - \underset{\prime}{w}} + \underset{\prime}{u}\underset{\prime}{w} \times \overline{x - \underset{\prime}{x}} = B$; and $uwxy - \underset{\prime}{u}\underset{\prime}{w}\underset{\prime}{x}\underset{\prime}{y} = yB + \underset{\prime}{u}\underset{\prime}{w}\underset{\prime}{x} \times \overline{y - \underset{\prime}{y}} = wxy \times \overline{u - \underset{\prime}{u}} + xy\underset{\prime}{u} \times \overline{w - \underset{\prime}{w}} + y\underset{\prime}{u}\underset{\prime}{w} \times \overline{x - \underset{\prime}{x}} + \underset{\prime}{u}\underset{\prime}{w}\underset{\prime}{x} \times \overline{y - \underset{\prime}{y}} = C$; &c.

It is evident therefore, that $uwx\,(n) - \underset{\prime}{u}\underset{\prime}{w}\underset{\prime}{x}\,(n)$ is $= wx\,(n-1) \times \overline{u - \underset{\prime}{u}} + xy\,(n-2) \times \underset{\prime}{u} \times \overline{w - \underset{\prime}{w}} + yz\,(n-3) \times \underset{\prime}{u}\underset{\prime}{w} \times \overline{x - \underset{\prime}{x}} + (n)$; and, confequently,

$$\overline{uwx\,(n) - \underset{\prime}{u}\underset{\prime}{w}\underset{\prime}{x}\,(n)} \div \overline{v - \underset{\prime}{v}} = wx\,(n-1) \times \frac{u - \underset{\prime}{u}}{v - \underset{\prime}{v}} + xy\,(n-2) \times \underset{\prime}{u} \times \frac{w - \underset{\prime}{w}}{v - \underset{\prime}{v}} + yz\,(n-3) \times \underset{\prime}{u}\underset{\prime}{w} \times \frac{x - \underset{\prime}{x}}{v - \underset{\prime}{v}} + (n).$$

EXAMPLE. Taking u, w, x, $\underset{\prime}{u}$, $\underset{\prime}{w}$, and $\underset{\prime}{x}$ equal to $v^{\frac{1}{3}}$, $\overline{a + v}|^{\frac{1}{2}}$, $\overline{b + v}|^{\frac{2}{3}}$, $\underset{\prime}{v}^{\frac{1}{3}}$, $\overline{a + \underset{\prime}{v}}|^{\frac{1}{2}}$, and $\overline{b + \underset{\prime}{v}}|^{\frac{2}{3}}$, refpectively; we have $\overline{v^{\frac{1}{3}} \times \overline{a + v}|^{\frac{1}{2}} \times \overline{b + v}|^{\frac{2}{3}} - \underset{\prime}{v}^{\frac{1}{3}} \times \overline{a + \underset{\prime}{v}}|^{\frac{1}{2}} \times \overline{b + \underset{\prime}{v}}|^{\frac{2}{3}}}$

$\div$

$\div \overline{v - v_{,}} = \overline{a+v}|^{\frac{1}{2}} \times \overline{b+v}|^{\frac{2}{3}} \times v^{-\frac{2}{3}} \times \dfrac{1}{1 + \overline{\frac{v_{,}}{v}}|^{\frac{1}{3}} + \overline{\frac{v_{,}}{v}}|^{\frac{2}{3}}} +$

$\overline{b+v}|^{\frac{2}{3}} \times v_{,}^{\frac{1}{3}} \times \overline{a+v}|^{-\frac{1}{2}} \times \dfrac{1}{1 + \overline{\frac{a+v_{,}}{a+v}}|^{\frac{1}{2}}} + v_{,}^{\frac{1}{3}} \times \overline{a+v_{,}}|^{\frac{1}{2}} \times$

$\overline{b+v}|^{-\frac{1}{3}} \times \dfrac{1 + \frac{b+v_{,}}{b+v}}{1 + \overline{\frac{b+v_{,}}{b+v}}|^{\frac{2}{3}} + \overline{\frac{b+v_{,}}{b+v}}|^{\frac{4}{3}}}$; $\overline{v^{\frac{1}{3}} - v_{,}^{\frac{1}{3}}} \div \overline{v - v_{,}}$ (by what is said above) being equal to $v^{-\frac{2}{3}} \times \dfrac{1}{1 + \overline{\frac{v_{,}}{v}}|^{\frac{1}{3}} + \overline{\frac{v_{,}}{v}}|^{\frac{2}{3}}}$,

$\overline{\overline{a+v}|^{\frac{1}{2}} - \overline{a+v_{,}}|^{\frac{1}{2}}} \div \overline{v - v_{,}}$ equal to $\overline{a+v}|^{-\frac{1}{2}} \times \dfrac{1}{1 + \overline{\frac{a+v_{,}}{a+v}}|^{\frac{1}{2}}}$,

and $\overline{\overline{b+v}|^{\frac{2}{3}} - \overline{b+v_{,}}|^{\frac{2}{3}}} \div \overline{v - v_{,}}$ equal to

$\overline{b+v}|^{-\frac{1}{3}} \times \dfrac{1 + \frac{b+v_{,}}{b+v}}{1 + \overline{\frac{b+v_{,}}{b+v}}|^{\frac{2}{3}} + \overline{\frac{b+v_{,}}{b+v}}|^{\frac{4}{3}}}$.

Corollary. When $v_{,}$ is equal to v; $u_{,}$, $w_{,}$, $x_{,}$, $y_{,}$, &c. will be respectively equal to u, w, x, y, &c. these last quantities being any functions of v. It is evident therefore, that $[v \perp uwx\,(n)]$ will be $= wxy\,(n-1) \times [v \perp u] + uxy\,(n-1) \times [u \perp w] + uwy\,(n-1) \times [v \perp x] + (n)$.

Example. Writing v^p, w^q, x^r, y^s; instead of u, w, x, and y respectively: we have $[v \perp u^p w^q x^r y^s] = w^q x^r y^s \times [v \perp u^p] +$

u^p

$\overset{p}{u}\overset{r}{x}\overset{s}{y} \times [v \dot{-} w^q] + u^p w^q y^s \times [v \dot{-} x^r] + u^p w^q x^r \times [v \dot{-} y^s] =$ $p w^q x^r y^s u^{p-1} \times [v \dot{-} u] + q u^p x^r y^s w^{q-1} \times [v \dot{-} w] + r u^p w^q y^s x^{r-1}$ $\times [v \dot{-} x] + s u^p w^q x^r y^{s-1} \times [v \dot{-} y]$; u, w, x, and y being any functions of v.

5.

Suppose $\overline{F - F_{,}} \div \overline{x - x_{,}}, = Q$, and $\overline{f - f_{,}} \div \overline{x - x_{,}} = q$.

Then, $F - F_{,}$ being $= Q \times \overline{x - x_{,}}$, and $f - f_{,} = q \times \overline{x - x_{,}}$,

$$\overline{F - F_{,}} \div \overline{f - f_{,}} \text{ is manifestly } = \frac{Q}{q}.$$

EXAMPLE. If F, f, $F_{,}$, and $f_{,}$ be equal to $\overline{ax + x^2}\big|^{\frac{2}{3}}$, $x^{\frac{1}{2}}$, $\overline{ax_{,} + x_{,}^2}\big|^{\frac{2}{3}}$, and $x_{,}^{\frac{1}{2}}$ respectively; $\overline{F - F_{,}} \div \overline{f - f_{,}}$ will, by what is done above, be

$$= \overline{ax + x^2}\big|^{-\frac{1}{3}} \times \frac{1 + \dfrac{ax_{,} + x_{,}^2}{ax + x^2}}{1 + \overline{\dfrac{ax_{,} + x_{,}^2}{ax + x^2}}\Big|^{\frac{2}{3}} + \overline{\dfrac{ax_{,} + x_{,}^2}{ax + x^2}}\Big|^{\frac{4}{3}}} \times \overline{a + x_{,} + x} \div \frac{1}{x^{\frac{1}{2}} + x_{,}^{\frac{1}{2}}}.$$

COROLLARY. F and f being any functions of x, the value of the quotient or fraction $\dfrac{F - F_{,}}{f - f_{,}}$, when $x_{,}$ is $= x$, (*i. e.* when both numerator and denominator vanish together,) is $= \dfrac{[x \dot{-} F]}{[x \dot{-} f]}$.

EXAMPLE.

Example. If F and f be equal to $\overline{a^2+x^2}\big|^{\frac{3}{2}}$ and $\overline{a+x}\big|^{\frac{3}{2}}$ respectively, the value of $\frac{F-F_{,}}{f-f_{,}}\left(=\frac{\overline{a^2+x^2}\big|^{\frac{3}{2}}-\overline{a^2+x_{,}^2}\big|^{\frac{3}{2}}}{\overline{a+x}\big|^{\frac{3}{2}}-\overline{a+x_{,}}\big|^{\frac{3}{2}}}\right)$, when $x_{,}$ is $= x$, is equal to $\frac{3x\times\overline{a^2+x^2}\big|^{\frac{1}{2}}}{\frac{3}{2}\times\overline{a+x}\big|^{\frac{1}{2}}} = 2x\times\overline{\frac{a^2+x^2}{a+x}}\Big|^{\frac{1}{2}}$.

6.

N and D being algebraic expressions so composed of x and other quantities, that each of those expressions becomes equal to *Nothing*, when x is equal to some certain quantity k; it is obvious, that $N_{,}$ and $D_{,}$ (the expressions which result by writing $x_{,}$ instead of x in N and D respectively) will both vanish when $x_{,}$ is equal to k. Therefore, if, in the quotient of $\overline{N-N_{,}}$ divided by $\overline{D-D_{,}}$, x be taken equal to k, the resulting expression will be equal to $\frac{N_{,}}{D_{,}}$, let $x_{,}$ be what it will. Now, by what is shewn above, the quotient of $\overline{N-N_{,}}$ divided by $\overline{D-D_{,}}$ $\left(=\frac{\overline{N-N_{,}}\div\overline{x-x_{,}}}{\overline{D-D_{,}}\div\overline{x-x_{,}}}\right)$ may always be readily assigned in terms which shall not vanish when x and $x_{,}$ are each equal to k, unless the value of that quotient be then $= 0$: consequently, by assigning such quotient, the value of $\frac{N}{D}$ will be had, as well when N and D are each equal to *Nothing*, as in any other case.

Example

EXAMPLE I. *To divide* $\sqrt{2a^2 + 2x^2} - 2a^{\frac{2}{3}}x^{\frac{1}{3}}$ *by* $x - a$.

The dividend and divisor both vanishing when x is equal to a, it is obvious, that, if, in the quotient of $\overline{\sqrt{2a^2 + 2x^2} - 2a^{\frac{2}{3}}x^{\frac{1}{3}}} - \overline{\sqrt{2a^2 + 2x_{,}^2} - 2a^{\frac{2}{3}}x_{,}^{\frac{1}{3}}}$ divided by $\overline{x - a} - \overline{x_{,} - a}$, $x_{,}$ be taken equal to a, the resulting expression will be equal to $\frac{\sqrt{2a^2 + 2x^2} - 2a^{\frac{2}{3}}x^{\frac{1}{3}}}{x - a}$, let x be what it will.

Now $\frac{\overline{\sqrt{2a^2 + 2x^2} - 2a^{\frac{2}{3}}x^{\frac{1}{3}}} - \overline{\sqrt{2a^2 + 2x_{,}^2} - 2a^{\frac{2}{3}}x_{,}^{\frac{1}{3}}}}{\overline{x - a} - \overline{x_{,} - a}}$ is manifestly

$= \frac{\sqrt{2a^2 + 2x^2} - \sqrt{2a^2 + 2x_{,}^2}}{x - x_{,}} - 2a^{\frac{2}{3}} \times \frac{x^{\frac{1}{3}} - x_{,}^{\frac{1}{3}}}{x - x_{,}}$; which, by what is done above, is $= \frac{2 \times \overline{x + x_{,}}}{\sqrt{2a^2 + 2x^2} + \sqrt{2a^2 + 2x_{,}^2}} - \frac{2a^{\frac{2}{3}}}{x^{\frac{2}{3}} + x^{\frac{1}{3}}x_{,}^{\frac{1}{3}} + x_{,}^{\frac{2}{3}}}$.

Consequently, by taking $x_{,}$ equal to a, we have

$$\frac{\sqrt{2a^2 + 2x^2} - 2a^{\frac{2}{3}}x^{\frac{1}{3}}}{x - a} = \frac{2 \times \overline{a + x}}{2a + \sqrt{2a^2 + 2x^2}} - \frac{2a^{\frac{2}{3}}}{a^{\frac{2}{3}} + a^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}}.$$

Whence it is evident, that, when x is equal to a, the quotient of $\overline{\sqrt{2a^2 + 2x^2} - 2a^{\frac{2}{3}}x^{\frac{1}{3}}} \div \overline{x - a}$ is $= \frac{1}{3}$.

EXAMPLE II. *Suppose* N *and* D *equal to* $\sqrt{2k^3x - x^4} - \sqrt{kx^3}$ *and* $2x - \overline{7k^3 + x^3}|^{\frac{1}{3}}$ *respectively.*

Then $\overline{\overline{\sqrt{2k^3x - x^4} - \sqrt{kx^3}} - \overline{\sqrt{2k^3x_{,} - x_{,}^4} - \sqrt{kx_{,}^3}}} \div \overline{x - x_{,}}$ being $= \overline{\sqrt{2k^3x - x^4} - \sqrt{2k^3x_{,} - x_{,}^4}} \div \overline{x - x_{,}} -$

$$\left(\sqrt{kx^3} - \sqrt{kx_{,}^3}\right) \div \left(x - x_{,}\right) = \frac{2k^3 - x^3 - x^2x_{,} - xx_{,}^2 - x_{,}^3}{\sqrt{2k^3x - x^4} + \sqrt{2k^3x_{,} - x_{,}^4}} - k^{\frac{1}{2}}x^{\frac{1}{2}} \times \frac{1 + \frac{x_{,}}{x} + \left(\frac{x_{,}}{x}\right)^{\frac{1}{2}}}{\left(1 + \frac{x_{,}}{x}\right)^{\frac{3}{2}}},$$

and

$$\left(2x - \left(7k^3 + x^3\right)^{\frac{1}{3}} - \left(2x_{,} - \left(7k^3 + x_{,}^3\right)^{\frac{1}{3}}\right)\right) \div \left(x - x_{,}\right) = \left(2x - 2x_{,}\right) \div \left(x - x_{,}\right) - \left(\left(7k^3 + x^3\right)^{\frac{1}{3}} - \left(7k^3 + x_{,}^3\right)^{\frac{1}{3}}\right)$$

$$\div \left(x - x_{,}\right) = 2 - \left(7k^3 + x^3\right)^{-\frac{2}{3}} \times \frac{x^2 + xx_{,} + x_{,}^2}{1 + \left(\frac{7k^3 + x_{,}^3}{7k^3 + x^3}\right)^{\frac{1}{3}} + \left(\frac{7k^3 + x_{,}^3}{7k^3 + x^3}\right)^{\frac{2}{3}}},$$

$$\frac{\left(N - N_{,}\right) \div \left(x - x_{,}\right)}{\left(D - D_{,}\right) \div \left(x - x_{,}\right)} \text{ will be } =$$

$$\frac{\dfrac{2k^3 - x^3 - x^2x_{,} - xx_{,}^2 - x_{,}^3}{\sqrt{2k^3x - x^4} + \sqrt{2k^3x_{,} - x_{,}^4}} - k^{\frac{1}{2}}x^{\frac{1}{2}} \times \dfrac{1 + \frac{x_{,}}{x} + \left(\frac{x_{,}}{x}\right)^{\frac{1}{2}}}{\left(1 + \frac{x_{,}}{x}\right)^{\frac{3}{2}}}}{2 - \dfrac{x^2 + xx_{,} + x_{,}^2}{\left(7k^3 + x^3\right)^{\frac{2}{3}} + \left(7k^3 + x^3\right)^{\frac{1}{3}} \times \left(7k^3 + x_{,}^3\right)^{\frac{1}{3}} + \left(7k^3 + x_{,}^3\right)^{\frac{2}{3}}}}.$$

From whence, by taking $x_{,}$ equal to k, we get $\frac{N}{D} =$

$$\frac{\dfrac{k^3 - x^3 - kx^2 - k^2x}{\sqrt{2k^3x - x^4} + k^2} - \left(kx\right)^{\frac{1}{2}} \times \dfrac{1 + \frac{k}{x} + \left(\frac{k}{x}\right)^{\frac{1}{2}}}{\left(1 + \frac{k}{x}\right)^{\frac{3}{2}}}}{2 - \dfrac{x^2 + kx + k^2}{\left(7k^3 + x^3\right)^{\frac{2}{3}} + 2k \times \left(7k^3 + x^3\right)^{\frac{1}{3}} + 4k^2}}.$$

Hence

Hence it is evident, that when x is $= k$,

$$\frac{\sqrt{2k^3x - x^4} - \sqrt{kx^3}}{2x - \overline{7k^3 + x^3}|^{\frac{1}{3}}} \text{ is } = -\frac{10}{7}k.$$

COROLLARY. Seeing it amounts to the ſame thing, to take $\underset{,}{x}$ equal to x, and afterwards x equal to k; as to take, at firſt, $\underset{,}{x}$ and x each equal to k: it is obvious, that the quotients or fractions $\frac{N}{D}$ and $\frac{[x \perp N]}{[x \perp D]}$ become equal to each other, when x, in each, is taken equal to k.

EXAMPLE I. Suppoſing N and D equal to $\sqrt{2r^3x - x^4} - \sqrt{rx^3}$ and $2x - \overline{7r^3 + x^3}|^{\frac{1}{3}}$ reſpectively; we have $[x \perp N] = \frac{r^3 - 2x^3}{\sqrt{2r^3x - x^4}} - \frac{3}{2}r^{\frac{1}{2}}x^{\frac{1}{2}}$, and $[x \perp D] = 2 - \frac{x^2}{\overline{7r^3 + x^3}|^{\frac{2}{3}}}$. It appears then, that the value of $\frac{[x \perp N]}{[x \perp D]}$, when x is taken equal to r, is $= -\frac{10r}{7}$; and the ſame is the value of $\frac{N}{D}$, when x is ſo taken; which agrees with what is done in the preceding Example.

EXAMPLE II. If it be required to find the value of the quotient of $u\sqrt{1 - \underset{,}{u}^2} - \underset{,}{u}\sqrt{1 - u^2}$ divided by $u - \underset{,}{u}$, when u is therein taken equal to $\underset{,}{u}$; we are to conſider $\underset{,}{u}$ as invariable: then, the ſpecial value of

$$\overline{\overline{u\sqrt{1 - \underset{,}{u}^2} - \underset{,}{u}\sqrt{1 - u^2}} - \overline{u\sqrt{1 - \underset{,,}{u}^2} - \underset{,,}{u}\sqrt{1 - u^2}}} \div \overline{\underset{,}{u} - \underset{,,}{u}}$$

being $= -\frac{u\underset{,}{u}}{\sqrt{1 - \underset{,}{u}^2}} - \sqrt{1 - u^2}$ and $\overline{\overline{u - \underset{,}{u}} - \overline{u - \underset{,,}{u}}} \div \overline{\underset{,}{u} - \underset{,,}{u}}$ being $= -1$, we ſhall have $\frac{u^2}{\sqrt{1 - u^2}} + \sqrt{1 - u^2}$ $\left(= \frac{1}{\sqrt{1 - u^2}}\right)$ for the required value of $\overline{u\sqrt{1 - \underset{,}{u}^2} - \underset{,}{u}\sqrt{1 - u^2}} \div \overline{u - \underset{,}{u}}$.

EXAMPLE III. Let s, c, and y be any functions of u; and let it be required to find the value of the quotient of

$$\overline{sy - y_{\prime} \times cu\sqrt{1-u_{\prime}^{2}} - cu_{\prime}\sqrt{1-u^{2}} + suu_{\prime} + s\sqrt{1-u^{2}} \times \sqrt{1-u_{\prime}^{2}}}$$

divided by $u - u_{\prime}$, when $u_{\prime}$ is therein taken equal to u.

Here we are to consider u, s, c, and y as invariable; and we may observe, that the proposed quotient is

$= s \times \overline{y - uu_{\prime}y_{\prime} - y_{\prime}\sqrt{1-u^{2}} \times \sqrt{1-u_{\prime}^{2}}} \div \overline{u - u_{\prime}} - cy_{\prime} \times \overline{u\sqrt{1-u_{\prime}^{2}} - u_{\prime}\sqrt{1-u^{2}}} \div \overline{u - u_{\prime}}$. Now the special value of $\overline{\overline{y - uu_{\prime}y_{\prime} - y_{\prime}\sqrt{1-u^{2}} \times \sqrt{1-u_{\prime}^{2}}} - \overline{y - uu_{\prime\prime}y_{\prime\prime} - y_{\prime\prime}\sqrt{1-u^{2}} \times \sqrt{1-u_{\prime\prime}^{2}}}}$ $\div \overline{u_{\prime} - u_{\prime\prime}}$ is $= - [u_{\prime} \perp y_{\prime}] \times \overline{uu_{\prime} + \sqrt{1-u^{2}} \times \sqrt{1-u_{\prime}^{2}}} - uy_{\prime} + \dfrac{u_{\prime}y_{\prime}\sqrt{1-u^{2}}}{\sqrt{1-u_{\prime}^{2}}}$, and $\overline{\overline{u - u_{\prime}} - \overline{u - u_{\prime\prime}}} \div \overline{u_{\prime} - u_{\prime\prime}}$ is $= -1$; moreover, the value of $\overline{u\sqrt{1-u_{\prime}^{2}} - u_{\prime}\sqrt{1-u^{2}}} \div \overline{u - u_{\prime}}$, when $u_{\prime}$ is $= u$, is equal to $\dfrac{1}{\sqrt{1-u^{2}}}$, by the preceding Example: therefore, $[u_{\prime} \perp y_{\prime}]$ being $= [u \perp y]$ when $u_{\prime}$ is $= u$, and $y_{\prime}$ being then $= y$; $s\,[u \perp y] - \dfrac{cy}{\sqrt{1-u^{2}}}$ is the required value of the quotient proposed.

EXAMPLE IV. Suppose y to be any function of x: Then, $[x \perp y] - [x \mid y] = [x \perp y] - \dfrac{y - y_{\prime}}{x - x_{\prime}}$ being $=$ $\dfrac{\overline{x - x_{\prime}} \times [x \perp y] - \overline{y - y_{\prime}}}{x - x_{\prime}}$, we have $\dfrac{[x \perp y] - [x \mid y]}{x - x_{\prime}} = \dfrac{\overline{x - x_{\prime}} \times [x \perp y] - \overline{y - y_{\prime}}}{\overline{x - x_{\prime}}\,|^{2}}$,

where

where both numerator and denominator vanish, when $x_{,}$ is $= x$ ($y_{,}$ being then $= y$).

Therefore, by our rule, the quotients or fractions $\frac{[x \perp y] - [x \mid y]}{x - x_{,}}$ and $\frac{-[x \perp y] + [x_{,} \perp y_{,}]}{-2 \times \overline{x - x_{,}}}$ are equal to each other, when $x_{,}$ in each, is taken equal to x. But both numerator and denominator of the fraction $\frac{-[x \perp y] + [x_{,} \perp y_{,}]}{-2 \times \overline{x - x_{,}}}$ vanish, when $x_{,}$ is equal to x. Therefore, applying our rule a second time, it appears that the quotients or fractions $\frac{[x \perp y] - [x \mid y]}{x - x_{,}}$ and $\frac{[x_{,} \perp\!\!\perp y_{,}]}{2}$ are equal to each other, when $x_{,}$ in each, is taken equal to x.

Consequently the value of the quotient or fraction $\frac{[x \perp y] - [x \mid y]}{x - x_{,}}$, when $x_{,}$ is taken $= x$, is equal to $\frac{1}{2} \times [x \perp\!\!\perp y]$; which, it is plain, is the value of $\frac{[x_{,} \perp\!\!\perp y_{,}]}{2}$ when $x_{,}$ is taken as just now mentioned.

7.

y, z, &c. being any functions of x; and F and f being algebraic expressions composed, in any manner, of all, or any of, the quantities x, y, z, &c. and any invariable quantities:

$$\text{If } F \text{ be always } = f,$$
$$F_{,} \text{ will be } = f_{,},$$
$$F - F_{,} = f - f_{,},$$
$$\overline{F - F_{,}} \div \overline{x - x_{,}} = \overline{f - f_{,}} \div \overline{x - x_{,}},$$
$$\text{and consequently } [x \perp F] = [x \perp f].$$

☞ The

☞ The deducing this laſt equation from the firſt, (and likewiſe every ſimilar operation,) I call *reſidual diviſion*:—For the ready performing of which, the Corollaries to the firſt four Articles may ſerve as Rules.

EXAMPLE I. If ax be $= by^m$, we ſhall, by reſidual diviſion, have

$$a = [x \dot{-} by^m] = bmy^{m-1} \times [x \dot{-} y];$$

$$\text{or } a \times [y \dot{-} x] = [y \dot{-} by^m] = bmy^{m-1};$$

$$\text{or } a \times [v \dot{-} x] = [v \dot{-} by^m] = bmy^{m-1} \times [v \dot{-} y],$$

v being any function of x or y.

EXAMPLE II. If y be $= a + bx - cz$, we ſhall have

$$[x \dot{-} y] = b - c \times [x \dot{-} z],$$

$$\text{or } \quad 1 = b\,[y \dot{-} x] - c\,[y \dot{-} z],$$

$$\text{or } [z \dot{-} y] = b\,[z \dot{-} x] - c.$$

EXAMPLE III. From the equation $ay^m = \overline{b^2 - x^2}|^n$, we get

$$amy^{m-1}[x \dot{-} y] = -2nx \times \overline{b^2 - x^2}|^{n-1};$$

$$\text{or } amy^{m-1} = -2nx\,[y \dot{-} x] \times \overline{b^2 - x^2}|^{n-1};$$

$$\text{or } amy^{m-1} \times [v \dot{-} y] = -2nx\,[v \dot{-} x] \times \overline{b^2 - x^2}|^{n-1},$$

v being any function of x or y.

EXAMPLE IV. From the equation $x = ay^m z^n$, we find

$$1 = amz^n y^{m-1} \times [x \dot{-} y] + any^m z^{n-1} \times [x \dot{-} z].$$

EXAMPLE V. If x be $= \frac{y}{z} + \sqrt{y^2 + z^2} = yz^{-1} + \overline{y^2 + z^2}|^{\frac{1}{2}}$, we ſhall find, by our reſidual diviſion,

$$1 = \frac{[x \dot{-} y]}{z} - \frac{y\,[x \dot{-} z]}{z^2} + \frac{y\,[x \dot{-} y] + z\,[x \dot{-} z]}{\overline{y^2 + z^2}|^{\frac{1}{2}}}.$$

EXAMPLE

EXAMPLE VI. $a^m + y^m$ being $= x^p \times \overline{a^n + x^n}\big|^q$, we get
$$my^{m-1}[x \dot{-} y] = px^{p-1} \times \overline{a^n + x^n}\big|^q + nqx^{n+p-1} \times \overline{a^n + x^n}\big|^{q-1}.$$

EXAMPLE VII. From the equation $ax^2y + bxz^2 = cy^2z^2$, we get
$$2axy + ax^2[x \dot{-} y] + bz^2 + 2bxz[x \dot{-} z] = 2cyz^2[x \dot{-} y] + 2cy^2z[x \dot{-} z].$$

EXAMPLE VIII. $ax^m + bx^py^q + cy^n$ being $= 0$, we have
$$amx^{m-1} + bpy^qx^{p-1} + bqx^py^{q-1}[x \dot{-} y] + cny^{n-1}[x \dot{-} y] = 0.$$

EXAMPLE IX. Suppoſe $x = ay^2[x \dot{-} y]$: then, by our diviſion, we find
$$1 = 2ay[x \dot{-} y]^2 + ay^2[x \ddot{-} y].$$

EXAMPLE X. Let $axy + bx^2[x \dot{-} y]$ be ſuppoſed $= 0$: then we ſhall have
$$ay + ax[x \dot{-} y] + 2bx[x \dot{-} y] + bx^2[x \ddot{-} y] = 0.$$

EXAMPLE XI. $ax[x \dot{-} z] + by[x \ddot{-} y] + c[x \dot{-} y]^2$ being $= 0$, we find
$$a[x \dot{-} z] + ax[x \ddot{-} z] + b[x \dot{-} y][x \ddot{-} y] + by[x \dddot{-} y] + 2c[x \dot{-} y][x \ddot{-} y] = 0.$$

EXAMPLE XII. Suppoſe $x^2 + axy + by^2 = 0$: then, by the diviſion ſo often mentioned, we have
$$2x + ay + ax[x \dot{-} y] + 2by[x \dot{-} y] = 0:$$
And from hence, by a ſimilar proceſs, we get
$$2 + 2a[x \dot{-} y] + ax[x \ddot{-} y] + 2b[x \dot{-} y]^2 + 2by[x \ddot{-} y] = 0.$$

EXAMPLE XIII. If y be $= ax^m$; $[x \dot{-} y]$ will be $= amx^{m-1}$, $[x \ddot{-} y] = am \,.\, \overline{m-1} \,.\, x^{m-2}$, and $[x \dddot{-} y] = am \,.\, \overline{m-1} \,.\, \overline{m-2} \,.\, x^{m-3}$, &c. as appears by the method of diviſion above-mentioned.

8.

8.

$\frac{z-z_{\prime}}{x-x_{\prime}}$ is manifestly $=\frac{z-z_{\prime}}{y-y_{\prime}} \div \frac{x-x_{\prime}}{y-y_{\prime}}$: it is obvious therefore, that, y and z being any functions of x,

$$[x \overset{\prime}{-} z] \text{ is } = \frac{[y \overset{\prime}{-} z]}{[y \overset{\prime}{-} x]}.$$

9.

$\frac{y-y_{\prime}}{x-x_{\prime}} \times \frac{x-x_{\prime}}{y-y_{\prime}}$ being $=1$, $[x \overset{\prime}{-} y] \times [y \overset{\prime}{-} x]$ is evidently $=1$,

$$\text{and } [x \overset{\prime}{-} y] = \frac{1}{[y \overset{\prime}{-} x]}.$$

From this last equation, we get, by residual division, $[x \overset{\prime\prime}{-} y] = \left[x \overset{\prime}{-} \frac{1}{[y \overset{\prime}{-} x]}\right]$, which (by what is shewn in the preceding Article) is $= \left[y \overset{\prime}{-} \frac{1}{[y \overset{\prime}{-} x]}\right] \times \frac{1}{[y \overset{\prime}{-} x]} = -\frac{[y \overset{\prime\prime}{-} x]}{[y \overset{\prime}{-} x]^3}$.

Moreover, from the equation $[x \overset{\prime\prime}{-} y] = -\frac{[y \overset{\prime\prime}{-} x]}{[y \overset{\prime}{-} x]^3}$, we find, by residual division, $[x \overset{\prime\prime\prime}{-} y] = -\left[x \overset{\prime}{-} \frac{[y \overset{\prime\prime}{-} x]}{[y \overset{\prime}{-} x]^3}\right] = -\left[y \overset{\prime}{-} \frac{[y \overset{\prime\prime}{-} x]}{[y \overset{\prime}{-} x]^3}\right] \times \frac{1}{[y \overset{\prime}{-} x]} = -\frac{[y \overset{\prime\prime\prime}{-} x]}{[y \overset{\prime}{-} x]^4} + \frac{3[y \overset{\prime\prime}{-} x]^2}{[y \overset{\prime}{-} x]^5}$, &c.

From what is done in this Chapter, it appears how the values of the quantities $[x \overset{\prime}{-} y]$, $[x \overset{\prime\prime}{-} y]$, &c. or $[y \overset{\prime}{-} x]$, $[y \overset{\prime\prime}{-} x]$, &c. may be obtained in terms of x and y, from an equation shewing the relation of the two quantities last mentioned, when no exponential quantity is concerned therein; which values we shall have frequent occasion to compute in the investigation of propositions by this Analysis. Rules for computing the values of the said quantities $[x \overset{\prime}{-} y]$, $[x \overset{\prime\prime}{-} y]$, &c. $[y \overset{\prime}{-} x]$, $[y \overset{\prime\prime}{-} x]$, &c. from equations containing exponentials, will be given in the next Chapter.

THE

RESIDUAL ANALYSIS.

CHAP. III.

Of Exponentials and Logarithms.

AS one variable quantity may be denoted by the algebraic expression x^n, or the like, where the Root only is variable whilst the Exponent remains invariable; so may another variable quantity be denoted by n^x, or the like, where the Exponent only is variable whilst the Root remains invariable. Therefore, having shewn how $x^n - x_{\prime}^{n}$, may be divided by $x - x_{\prime}$, it will now be proper to shew how $n^x - n^{x_{\prime}}$ may also be divided by the same divisor $(x - x_{\prime})$.—In doing which, we shall first assign the value of n^x in a certain series of terms of n and x, wherein the exponents of the several powers of these quantities shall be invariable: by help of which series we shall be enabled readily to obtain the desired quotient of $n^x - n^{x_{\prime}}$ divided by $x - x_{\prime}$.

We ſhall then ſhew how, by means of that quotient, the value of n^x may be aſſigned in another ſeries of terms of n and x, in which the exponents ſhall be invariable; and likewiſe how, from the equation $n^x = v$, the value of x may be aſſigned in a ſeries of terms of n and v, with invariable exponents.—From which ſeries many uſeful concluſions relating to Exponentials and Logarithms will be eaſily deduced.

I.

Aſſuming $\mathrm{x}^{\frac{\mathrm{m}}{\mathrm{r}}} = 1 + \mathrm{b}\,.\,\overline{\mathrm{x}-1} + \mathrm{c}\,.\,\overline{\mathrm{x}-1}|^2 + \mathrm{d}\,.\,\overline{\mathrm{x}-1}|^3$ &c. *it is propoſed to find, in terms of* m *and* r, *the coefficients* b, c, d, *&c.*

Writing (in the aſſumed equation) y inſtead of x, we have

$$y^{\frac{m}{r}} = 1 + b\,.\,\overline{y-1} + c\,.\,\overline{y-1}|^2 + d\,.\,\overline{y-1}|^3 \text{ \&c.}$$

and by ſubtraction,

$$x^{\frac{m}{r}} - y^{\frac{m}{r}} = b\,.\,\overline{x-y} + c\,.\,\overline{\overline{x-1}|^2 - \overline{y-1}|^2} +$$

$$d\,.\,\overline{\overline{x-1}|^3 - \overline{y-1}|^3} \text{ \&c.}$$

If, now, we divide by the reſidual $x - y$, we ſhall get

$$x^{\frac{m}{r}-1} \times \frac{1 + \frac{y}{x} \quad + \overline{\frac{y}{x}}|^2 \qquad (m)}{1 + \overline{\frac{y}{x}}|^{\frac{m}{r}} + \overline{\frac{y}{x}}|^{\frac{2m}{r}} \qquad (r)} =$$

$$b + c\,.\,\overline{\overline{x-1} + \overline{y-1}} + d\,.\,\overline{\overline{x-1}|^2 + \overline{x-1} \times \overline{y-1} + \overline{y-1}|^2}$$

&c. which equation muſt hold true let y be what it will: From whence, by taking y equal to x, we find

$$\frac{m}{r} \times x^{\frac{m}{r}-1} = b + 2c\,.\,\overline{x-1} + 3d\,.\,\overline{x-1}|^2 \text{ \&c.}$$

Conſe-

Consequently, multiplying one side by x and the other by $1 + \overline{x-1}$, we have $\frac{m}{r} \times x^{\frac{m}{r}}$, or its equal

$$\frac{m}{r} + \frac{m}{r}b.\overline{x-1} + \frac{m}{r}c.\overline{x-1}|^{2} + \frac{m}{r}d.\overline{x-1}|^{3} \&c.$$

$$= b + \left.\begin{matrix}2c\\ b\end{matrix}\right\}\overline{x-1} + \left.\begin{matrix}3d\\ 2c\end{matrix}\right\}\overline{x-1}|^{2} + \left.\begin{matrix}4e\\ 3d\end{matrix}\right\}\overline{x-1}|^{3} \&c.$$

Hence, by equating the homologous terms, it appears, that b is $= \frac{m}{r}$, $c = \frac{m-r}{2r}.b$, $d = \frac{m-2r}{3r}.c$, $e = \frac{m-3r}{4r}.d$, &c.

In like manner, if we assume

$$\frac{1}{x^{\frac{m}{r}}} = 1 + b.\overline{x-1} + c.\overline{x-1}|^{2} + d.\overline{x-1}|^{3} \&c.$$

we shall find $b = -\frac{m}{r}$, $c = -\frac{m+r}{2r}.b$, $d = -\frac{m+2r}{3r}.c$, &c.

Consequently, $\frac{1}{x^{\frac{m}{r}}}$ being $= x^{-\frac{m}{r}}$, it is manifest, that, whether the exponent $\frac{m}{r}$ be positive or negative, $x^{\frac{m}{r}}$ is $= 1 + \frac{m}{r}.\overline{x-1} + \frac{m}{r}.\frac{m-r}{2r}.\overline{x-1}|^{2} + \frac{m}{r}.\frac{m-r}{2r}.\frac{m-2r}{3r}.\overline{x-1}|^{3}$ &c.

Hence, by writing n instead of x, and x instead of $\frac{m}{r}$, we have $n^{x} = 1 + x.\overline{n-1} + x.\frac{x-1}{2}.\overline{n-1}|^{2} + x.\frac{x-1}{2}.\frac{x-2}{3}.\overline{n-1}|^{3}$ &c. x being any number whatever.

And, by substituting, in this last equation, $\frac{a+z}{a}$ instead of n, it appears that $\overline{a+z}|^{x}$ is $= a^{x} + x.a^{x-1}z + x.\frac{x-1}{2}.a^{x-2}z^{2} + x.\frac{x-1}{2}.\frac{x-2}{3}.a^{x-3}z^{3}$ &c. which is the Binomial Theorem *.

2. The

* The same conclusion, by referring to what is proved in the preceding Chapter, may be obtained in the following brief manner; but I presumed,

that,

2.

The expression $\frac{n^{x} - n^{\underset{,}{x}}}{x - \underset{,}{x}}$ being $= n^{\underset{,}{x}} \times \frac{n^{x-\underset{,}{x}} - 1}{x - \underset{,}{x}}$, which by the preceding Article is

$$= n^{\underset{,}{x}} \times \overline{n-1} + \frac{x-\underset{,}{x}-1}{2}.\overline{n-1}|^{2} + \frac{x-\underset{,}{x}-1}{2}.\frac{x-\underset{,}{x}-2}{3}.\overline{n-1}|^{3}$$

&c. it is obvious, that, when $\underset{,}{x}$ is equal to x, the value of the quotient of $n^{x} - n^{\underset{,}{x}}$ divided by $x - \underset{,}{x}$ is

$$= n^{x} \times \overline{\overline{n-1} - \frac{\overline{n-1}|^{2}}{2} + \frac{\overline{n-1}|^{3}}{3} - \frac{\overline{n-1}|^{4}}{4}\ \&c.} = g \times n^{x},$$

g being put for the series $\overline{n-1} - \frac{\overline{n-1}|^{2}}{2} + \frac{\overline{n-1}|^{3}}{3} -$ &c. which converges when n is a positive number less than 2.

Writing $\frac{1}{n}$ instead of n, we get

$$\textit{spec. val. of } \overline{\frac{1}{n^{x}} - \frac{1}{n^{\underset{,}{x}}}} \div \overline{x - \underset{,}{x}} = -\frac{1}{n^{2x}} \times \textit{spec. val. of } \frac{n^{x} - n^{\underset{,}{x}}}{x - \underset{,}{x}}$$

$$= \frac{1}{n^{x}} \times \overline{\frac{1-n}{n} - \frac{1}{2}.\overline{\frac{1-n}{n}}|^{2} + \frac{1}{3}.\overline{\frac{1-n}{n}}|^{3} - \&c.}$$

---

that, upon the first application of what is there taught, it would not be amiss to give a more explicit process.

The brief investigation is this.

Assuming $\overline{1+v}|^{x} = 1 + bv + cv^{2} + dv^{3}$ &c.

we get, by residual division, ($v - \underset{,}{v}$ being the divisor,)

$$x \times \overline{1+v}|^{x-1} = b + 2cv + 3dv^{2}\ \&c.$$

Consequently, multiplying by $1 + v$, we have

$$x \times \overline{1+v}|^{x}, \text{ or its equal } x + xbv + xcv^{2} + xdv^{3}\ \&c.$$
$$= b + \left.\begin{matrix}2c\\ b\end{matrix}\right\} v + \left.\begin{matrix}3d\\ 2c\end{matrix}\right\} v^{2} + \left.\begin{matrix}4e\\ 3d\end{matrix}\right\} v^{3}\ \&c.$$

From whence, by equating the homologous terms, b, c, d, &c. will be found equal to x, $\frac{x.\overline{x-1}}{2}$, $\frac{x.\overline{x-1}.\overline{x-2}}{2.3}$, &c. respectively: And then the value of $\overline{a+z}|^{x}$ may be obtained by substituting $\frac{z}{a}$ instead of v.

From

From whence we find, that the *spec. val.* of $\overline{n^{x} - n^{x_{,}}} \div \overline{x - x_{,}}$ is also $= n^{x} \times \overline{\frac{n-1}{n} + \frac{1}{2} \cdot \left|\frac{n-1}{n}\right|^{2} + \frac{1}{3} \cdot \left|\frac{n-1}{n}\right|^{3} \&c.} = g \times n^{x}$, g being now put for the series $\frac{n-1}{n} + \frac{1}{2} \cdot \left|\frac{n-1}{n}\right|^{2}$ &c. which converges when n is any positive number greater than $\frac{1}{2}$.

3.

Assuming $n^{x} = 1 + Ax + Bx^{2} + Cx^{3}$ &c. *it is proposed to find, in terms of* n, *the coefficients* A, B, C, &c.

$$n^{x_{,}} \text{ being } = 1 + Ax_{,} + Bx_{,}^{2} + Cx_{,}^{3} \ \&c.$$

we have, by subtraction,

$$n^{x} - n^{x_{,}} = A . \overline{x - x_{,}} + B . \overline{x^{2} - x_{,}^{2}} + C . \overline{x^{3} - x_{,}^{3}} \ \&c.$$

and from hence, by division,

$$\frac{n^{x} - n^{x_{,}}}{x - x_{,}} = A + B . \overline{x + x_{,}} + C . \overline{x^{2} + xx_{,} + x_{,}^{2}} \ \&c.$$

Now, the quotient of $n^{x} - n^{x_{,}}$ divided by $x - x_{,}$ being, by the last Article, equal to $g \times n^{x}$ when $x_{,}$ is $= x$, it follows, that

$$g \times n^{x}, \text{ or its equal } g + gAx + gBx^{2} + gCx^{3} \ \&c.$$
$$\text{is } = A + 2Bx + 3Cx^{2} + 4Dx^{3} \ \&c.$$

From whence, by comparing the homologous terms, A is found $= g$, $B = \frac{gA}{2}$, $C = \frac{gB}{3}$, $D = \frac{gC}{4}$, &c.

Therefore n^{x} is $= 1 + gx + \frac{g^{2}x^{2}}{2} + \frac{g^{3}x^{3}}{2.3} + \frac{g^{4}x^{4}}{2.3.4}$ &c.

4.

From the equation $n^{x} = v$, *it is now proposed to find* x *in terms of* n *and* v.

If

If x be assumed $= m\times\overline{\overline{v-1}+A.\overline{v-1}|^{2}+B.\overline{v-1}|^{3}\ \&c.}$ and $n^{x_{,}} = v_{,}$, (m, A, B, &c. being supposed independent of v,) we shall have

$$x_{,} = m\times\overline{\overline{v_{,}-1}+A.\overline{v_{,}-1}|^{2}+B.\overline{v_{,}-1}|^{3}\ \&c.}$$

and, by subtraction,

$$x - x_{,} = m\times\overline{\overline{v-v_{,}}+A.\overline{\overline{v-1}|^{2}-\overline{v_{,}-1}|^{2}}+B.\overline{\overline{v-1}|^{3}-\overline{v_{,}-1}|^{3}}\ \&c.}$$

From which last equation, by dividing one side by $n^{x} - n^{x_{,}}$ and the other by $v - v_{,}$ ($=\overline{v-1}-\overline{v_{,}-1}$), we get

$$\frac{x-x_{,}}{n^{x}-n^{x_{,}}} = m\times\overline{1+A.\overline{\overline{v-1}+\overline{v_{,}-1}}+B.\overline{\overline{v-1}|^{2}+\overline{v-1}\times\overline{v_{,}-1}+\overline{v_{,}-1}|^{2}}\ \&c.}$$

Hence, by taking $x_{,}$ equal to x, and $v_{,}$ equal to v, we have

$$\frac{1}{g\times n^{x}} = m\times\overline{1+2A.\overline{v-1}+3B.\overline{v-1}|^{2}+4C.\overline{v-1}|^{3}\ \&c.}$$

$g\times n^{x}$, by Art. 2, being the value of the quotient of $n^{x}-n^{x_{,}}$ divided by $x-x_{,}$, when $x_{,}$ is therein taken equal to x.

But $\frac{1}{n^{x}}$ is $=\frac{1}{v}=\frac{1}{1+\overline{v-1}}$, which, by division, is found equal to $1-\overline{v-1}+\overline{v-1}|^{2}-\overline{v-1}|^{3}+\overline{v-1}|^{4}-$ &c.

It is evident therefore, that

$$\frac{1}{g}\times\overline{1-\overline{v-1}+\overline{v-1}|^{2}-\overline{v-1}|^{3}+\overline{v-1}|^{4}-\&c.}$$

is $= m\times\overline{1+2A.\overline{v-1}+3B.\overline{v-1}|^{2}+4C.\overline{v-1}|^{3}+5D.\overline{v-1}|^{4}\ \&c.}$

From whence, by comparing the homologous terms, we find $m=\frac{1}{g}$, $A=-\frac{1}{2}$, $B=\frac{1}{3}$, $C=-\frac{1}{4}$, $D=\frac{1}{5}$, &c. and consequently $x=\frac{1}{g}\times\overline{\overline{v-1}-\frac{\overline{v-1}|^{2}}{2}+\frac{\overline{v-1}|^{3}}{3}-\frac{\overline{v-1}|^{4}}{4}+\&c.}$

5. By

5.

By putting $1+u$ equal to $\overset{x}{n}=v$, we have, by Article 3,

$$1+u=1+gx+\frac{g^2x^2}{2}+\frac{g^3x^3}{2\cdot 3}+\frac{g^4x^4}{2\cdot 3\cdot 4}\ \&c.$$

and from the equation

$$x=\frac{1}{g}\times\overline{v-1-\frac{\overline{v-1}^{\,2}}{2}+\frac{\overline{v-1}^{\,3}}{3}-\&c.}$$

$$x=\frac{1}{g}\times\overline{u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}+\&c.}$$

where g is $=\overline{n-1}-\frac{\overline{n-1}^{\,2}}{2}+\frac{\overline{n-1}^{\,3}}{3}-\frac{\overline{n-1}^{\,4}}{4}+\&c.$

$$\text{or } \frac{n-1}{n}+\frac{1}{2}\cdot\overline{\frac{n-1}{n}}^{\,2}+\frac{1}{3}\cdot\overline{\frac{n-1}{n}}^{\,3}\ \&c.$$

COROLLARY I. *Logarithms* being an artificial Set of numbers correfponding to another Set of numbers in fuch a manner, that the fum of the logarithms of any two numbers is equal to the logarithm of the product of thofe two numbers *; and the fum of the exponents of any two powers of any given quantity being equal to the exponent of that power (of the fame given quantity) which is produced by multiplying thofe two powers together; it is obvious that the exponents of the powers of any given quantity are logarithms of the values of thofe powers refpectively: For inftance, a, b, c, d, &c. are logarithms of $\overset{a}{n}$, $\overset{b}{n}$, $\overset{c}{n}$, $\overset{d}{n}$, &c. refpectively.

It follows therefore, from what is faid above, that

$$\frac{1}{g}\times\overline{v-1-\frac{\overline{v-1}^{\,2}}{2}+\frac{\overline{v-1}^{\,3}}{3}-\&c.}\quad \frac{1}{g}\times\overline{u-\frac{u^2}{2}+\frac{u^3}{3}-\&c.}$$

and $\frac{1}{g}\times\overline{w-\frac{w^2}{2}+\frac{w^3}{3}-\&c.}$ are logarithms of v, $1+u$,

* From this property of logarithms, it plainly follows, that, b, c, p, and N being any numbers whatever, the Log. of b — the Log. of c is $=$ the Log. of $\frac{b}{c}$, and $p\times$ Log. of N $=$ Log. of N^p.

and

and $1+w$ respectively, in the same Set, g being of any determinate value whatever.

From the equation, Log. of $\overline{1+u} = \frac{1}{g} \times \overline{u - \frac{u^2}{2} + \frac{u^3}{3} - \&c.}$ we, by writing $\frac{1}{u}$ instead of u, have Log. of $1 + \frac{1}{u}$ (= Log. of $\overline{1+u}$ — Log. of u) $= \frac{1}{g} \times \overline{u^{-1} - \frac{u^{-2}}{2} + \frac{u^{-3}}{3} - \&c.}$ and consequently

$$\text{Log. of } \overline{1+u} = \text{Log. of } u + \frac{1}{g} \times \overline{u^{-1} - \frac{u^{-2}}{2} + \frac{u^{-3}}{3} - \&c.}$$

which last series converges when the series in the former value of the Log. of $\overline{1+u}$ does not converge.

So from the equat. Log. of $v = \frac{1}{g} \times \overline{\overline{v-1} - \frac{\overline{v-1}|^2}{2} + \frac{\overline{v-1}|^3}{3} - \&c.}$ (where v must be a positive number less than 2,) we, by substituting $\frac{1}{v}$ instead of v, find

$$\text{Log. of } v = \frac{1}{g} \times \overline{\frac{v-1}{v} + \frac{1}{2} \cdot \overline{\frac{v-1}{v}}\Big|^2 + \frac{1}{3} \cdot \overline{\frac{v-1}{v}}\Big|^3} \ \&c.$$

where v may be any positive number greater than $\frac{1}{2}$.

Now, as in these logarithmic expressions the value of $\frac{1}{g}$ may be taken at pleasure, so that in any one Set it be always the same; it is plain, that, to any Set of numbers, we may adapt as many Sets of logarithms as we please, by taking $\frac{1}{g}$ of different values in different Sets; and the logarithm of any number, in any one Set of logarithms, being to the logarithm of the same number, in any other Set of logarithms, as the value of the factor $\frac{1}{g}$ in the former Set is to the value of $\frac{1}{g}$ in the latter, that factor is called the *Modulus* of the Set.

SCHOLIUM. *Lord* NAPIER, the Inventor of Logarithms, first gave a Set, whereof the modulus is Unity :—In which Set the logarithm

logarithm of 10 is 2.302585 *:—Of this Set are the logarithms now called *Napier's logarithms* †. And he afterwards, with the aſſiſtance of Mr. BRIGGS, (then *Savilian* Profeſſor of Geometry, at *Oxford*,) undertook to compute a ſecond Set of logarithms, wherein the logarithm of 10 is 1; (which Set he found would be more convenient for trigonometrical calculations than his former;) but, Lord NAPIER dying before they had finiſhed their deſign, the work was completed by Mr. BRIGGS, and the logarithms of this ſecond Set (which are the common tabular logarithms) are called *Briggs's logarithms*. The modulus of this ſecond Set is $\frac{1}{2.302585} = 0.4342944$: for, by what is ſaid above, the logarithm of 10, which in this Set is 1, is $= \frac{1}{g} \times 2.302585$; and, conſequently, $\frac{1}{g} = \frac{1}{2.302585}$.

COROLLARY II. From what is ſaid above, it plainly appears, that if, in any Set of logarithms whoſe modulus is $\frac{1}{g}$, x be the logarithm of any number whatſoever, that number will be equal to

$$1 + gx + \frac{g^2x^2}{2} + \frac{g^3x^3}{2 \cdot 3} + \frac{g^4x^4}{2 \cdot 3 \cdot 4} \text{ \&c.} = n^x.$$

COROLLARY III. By taking, in the laſt Corollary, x equal to $\frac{1}{g}$, it appears that, in any Set of logarithms, the modulus is always the logarithm of $1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4}$ &c. ($= 2.7182818$); or, which is the ſame thing, of the ratio of the ſum of that Series to Unity: which ratio is therefore, by Mr. COTES, called the *modular ratio*.

And, $-x$ being the logarithm of the reciprocal of any number whoſe logarithm is x; it appears by taking, in the ſame Corollary, x equal to $-\frac{1}{g}$, that the modular ratio is that of Unity to the ſum of the ſeries $1 - 1 + \frac{1}{2} - \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} -$ &c. ($= 0.3678794$).

* See Scholium to Article 6.

† Theſe are alſo called *Hyperbolic logarithms*, for a reaſon that will appear in a ſubſequent Chapter.

Corollary IV. It is evident then, that, in *Napier's* Set, (whereof the modulus is 1,) 2.7182818 is the number whose logarithm is 1; and that 0.3678794 is the number whose logarithm is -1: Also, that, n being $= 2.7182818$,

$$n^x \text{ is } = 1 + x + \frac{x^2}{2} + \frac{x^3}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4} \text{ \&c.}$$

6.

Napier's Log. of $\overline{1+x}$ being, by Art. 5, $= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} +$ &c. it appears, by writing $-x$ instead of x, that *Napier's Log.* of $\overline{1-x}$ is $= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$ &c. and, by subtraction, that *Nap. Log.* of $\overline{1+x}$ — *Nap. Log.* of $\overline{1-x}$ (= *Nap. Log.* of $\frac{1+x}{1-x}$) is equal to $2 \times \overline{x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} \text{ \&c.}}$

By this theorem the logarithms of small numbers may be easily computed.

Example I. *To find Napier's logarithm of* 2.

Supposing $\frac{1+x}{1-x} = 2$, x will from thence be found $= \frac{1}{3}$, and therefore *Nap. Log.* of 2 is $= 2 \times \overline{\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} \text{ \&c.}}$ $= 0.69314718$.

Example II. *To find Napier's logarithm of* $\frac{5}{4}$.

From the equation $\frac{1+x}{1-x} = \frac{5}{4}$ we find $x = \frac{1}{9}$; and, consequently, *Nap. Log.* of $\frac{5}{4} = 2 \times \overline{\frac{1}{1 \cdot 9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} \text{ \&c.}}$ $= 0.22314355$.

Scholium. To find the logarithm of a great number, by the same theorem, we must find the logarithms of such small numbers as, being multiplied together, shall produce that great number; and then, the sum of the logarithms of any numbers being equal to the logarithm of the product of those numbers, we shall, by adding those logarithms together, find the logarithm required.

Example.

EXAMPLE. *To find Napier's logarithm of* 10.

It is obvious, that 10 is $= 8 \times \frac{5}{4} = 2 \times 2 \times 2 \times \frac{5}{4}$; therefore, by what is just now said, *Napier's logarithm* of 2 being computed, and likewise his logarithm of $\frac{5}{4}$, the sum of this last logarithm added to three times the former will be $(= 2.30258509)$ the required logarithm of 10.

7.

Other theorems, more convenient for computing the logarithms of some certain numbers, may be deduced from the above; of which the following are examples.

EXAMPLE I. *To find the logarithm of* n, *having the logarithms of* n — 1 *and* n + 1 *given.*

Supposing $\frac{n^2}{\overline{n-1} \times \overline{n+1}}$, or its equal $\frac{n^2}{n^2 - 1} = \frac{1+x}{1-x}$, we from thence find $x = \frac{1}{2n^2 - 1}$. Therefore, by the theorem in the last Article, *Nap. Log.* of $\frac{n^2}{n^2 - 1}$ (= 2 × *Nap. Log.* of n — *Nap. Log.* of $\overline{n-1}$ — *Nap. Log.* of $\overline{n+1}$) is =

$$2 \times \overline{\frac{1}{2n^2 - 1} + \frac{1}{3 \cdot \overline{2n^2 - 1}|^3} + \frac{1}{5 \cdot \overline{2n^2 - 1}|^5} \;\&c.}$$

and, consequently,

$$\textit{Nap. Log. of } n = \frac{\textit{Nap. Log. of } \overline{n-1} + \textit{Nap. Log. of } \overline{n+1}}{2} + \frac{1}{2n^2 - 1} + \frac{1}{3 \cdot \overline{2n^2 - 1}|^3} + \frac{1}{5 \cdot \overline{2n^2 - 1}|^5} \;\&c.$$

This theorem is very convenient for computing the logarithm of any large prime number.

EXAMPLE II. *To find the logarithm of a large number* n, *having the logarithms of* n — x *and* n + x *given.*

The Log. of $1 - x$ being $= - \text{Mod.} \times \overline{x + \frac{x^2}{2} + \frac{x^3}{3}\ \&c.}$ the Log. of $\frac{1}{1-x}$ is $= \text{Mod.} \times \overline{x + \frac{x^2}{2} + \frac{x^3}{3}}\ \&c.$ which (the Log. of $\frac{1+x}{1-x}$ being $= 2 \times \text{Mod.} \times \overline{x + \frac{x^3}{3} + \frac{x^5}{5}}\ \&c.$) is $= \frac{1}{2} \times$ the Log. of $\overline{\frac{1+x}{1-x}} \times \frac{x + \frac{x^2}{2} + \frac{x^3}{3}\ \&c.}{x + \frac{x^3}{3} + \frac{x^5}{5}\ \&c.} = \frac{1}{2} \times$ Log. of $\overline{\frac{1+x}{1-x}} \times \overline{1 + \frac{x}{2} + \frac{x^3}{12} + \frac{7x^5}{180}}\ \&c.$ Hence, by substituting $\frac{x}{n}$ instead of x, we have Log. of $\frac{n}{n-x}$ (= Log. of n — Log. of $\overline{n-x}$) $= \frac{1}{2} \times$ Log. of $\overline{\frac{n+x}{n-x}} \times \overline{1 + \frac{x}{2n} + \frac{x^3}{12n^3} + \frac{x^5}{180n^5}}\ \&c.$ Therefore, the Log. of $\frac{n+x}{n-x}$ being = Log. of $\overline{n+x}$ — Log. of $\overline{n-x}$, the Log. of n is $= \frac{1}{2} \times$ Log. of $\overline{n+x} + \frac{1}{2} \times$ Log. of $\overline{n-x} +$ $\frac{\text{Log. of } \overline{n+x} - \text{Log. of } \overline{n-x}}{2} \times \overline{\frac{x}{2n} + \frac{x^3}{12n^3} + \frac{x^5}{180n^5}}\ \&c.$

This last theorem is very useful in enlarging a table of logarithms.

8.

It may sometimes be of use to observe, that, when v is a very large number, the Log. of $1 + \frac{1}{v}$ will be $= \frac{1}{v}$ nearly, the value of the series $\frac{v^{-2}}{2} - \frac{v^{-3}}{3} + \frac{v^{-4}}{4} -$ &c. (See Art. 5. Cor. 1.) being then so very small that it may be neglected. Therefore, v being as just now mentioned, if $\overline{1 + \frac{1}{v}}\Big|^{u}$ be = N, the Log. of N (= $u \times$ Log. of $\overline{1 + \frac{1}{v}}$) will be nearly equal to $\frac{u}{v}$.

9. By

9.

By Art. 2. $[x \dot{-} n^x]$ is $= gn^x$, g being (by that Article, and Cor. 1. Art. 5.) $=$ *Nap. Log.* of n.

But in *Napier*'s Set, 2.7182818 is the number whoſe logarithm is 1, as appears by Cor. 4. Art. 5.

Therefore if n be $= 2.7182818$ (*i. e.* if the ratio of n to 1 be the modular ratio) g will here be $= 1$; and, in that caſe, $[x \dot{-} n^x]$ is $= n^x$; or, v being $= n^x$, $[x \dot{-} v]$ is $= v$.

Conſequently $[x \dot{-} v] \times [v \dot{-} x]$ being $= 1$, (by Chap. 2. Art. 9.) $[v \dot{-} x]$ is $= \frac{1}{v}$, where x is *Nap. Log.* of v.

Moreover, by Chap. 2. Art. 8. $[v \dot{-} x]$ is equal to $\frac{[u \dot{-} x]}{[u \dot{-} v]}$: this laſt expreſſion muſt therefore be $= \frac{1}{v}$; and, conſequently,

$$[u \dot{-} x] = \frac{[u \dot{-} v]}{v},$$

u being any function of v, or x.

☞ For brevity ſake, we ſhall, in future, put $L : x$, $L : \overline{a + bx}$, $L : \overline{x + \sqrt{1 + x^2}}\,\big|^p$, &c. to denote *Nap. Log.* of x, *Nap. Log.* of $\overline{a + bx}$, *Nap. Log.* of $\overline{x + \sqrt{1 + x^2}}\,\big|^p$, &c. reſpectively.

EXAMPLE I. *Suppoſe* $v = 1 + z$: then we have

$$[u \dot{-} L : \overline{1 + z}] = \frac{[u \dot{-} z]}{1 + z}.$$

EXAMPLE II. *Suppoſing* $v = a + bz^m$, we have

$$[u \dot{-} L : \overline{a + bz^m}] = \frac{[u \dot{-} \overline{a + bz^m}]}{a + bz^m} = \frac{bmz^{m-1}[u \dot{-} z]}{a + bz^m}.$$

EXAMPLE III. *Let v be ſuppoſed* $= a + z + \sqrt{2az + z^2}$.

Then we have $[u \dot{-} L : \overline{a + z + \sqrt{2az + z^2}}] = \frac{[u \dot{-} z]}{\sqrt{2az + z^2}}$,

$[u \dot{-}$

$[u \dot{-} \overline{a+z+\sqrt{2az+z^2}}]$ being $= \frac{a+z+\sqrt{2az+z^2}}{\sqrt{2az+z^2}} \times [u \dot{-} z]$.

EXAMPLE IV. $[u \dot{-} \text{L} : \overline{z^n + \sqrt{a^2 + z^{2n}}}]$ is $= \frac{nz^{n-1}[u \dot{-} z]}{\sqrt{a^2 + z^{2n}}}$,

$[u \dot{-} \overline{z^n + \sqrt{a^2+z^{2n}}}]$ being $= \overline{z^n + \sqrt{a^2+z^{2n}}} \times \frac{nz^{n-1}[u \dot{-} z]}{\sqrt{a^2+z^{2n}}}$.

EXAMPLE V. *v being supposed* $= \frac{a+z}{a-z}$, we have

$$[u \dot{-} \text{L} : \frac{a+z}{a-z}] = \frac{2a[u \dot{-} z]}{\overline{a-z}|^2} \div \frac{a+z}{a-z} = \frac{2a[u \dot{-} z]}{a^2 - z^2}.$$

EXAMPLE VI. *Taking* $v = \frac{a - \sqrt{a^2+z^2}}{a + \sqrt{a^2+z^2}}$, we find

$$[u \dot{-} \text{L} : \frac{a - \sqrt{a^2+z^2}}{a + \sqrt{a^2+z^2}}] = \frac{2a[u \dot{-} z]}{z\sqrt{a^2+z^2}}.$$

10.

By the last Article we have $[x \dot{-} \text{L} : r^v] = \frac{[x \dot{-} r^v]}{r^v}$. Now $\text{L} : r^v$ being $= v \times \text{L} : r$, $[x \dot{-} \text{L} : r^v]$ is $= [x \dot{-} v \times \text{L} : r] = [x \dot{-} v] \times \text{L} : r$, v being any function of the variable quantity x, and r invariable: Therefore $\frac{[x \dot{-} r^v]}{r^v}$ is then $= [x \dot{-} v] \times \text{L} : r$; and, consequently, $[x \dot{-} r^v] = r^v \times [x \dot{-} v] \times \text{L} : r$.

But if both r and v be functions of x, $[x \dot{-} v \times \text{L} : r]$ $(= [x \dot{-} \text{L} : r^v] = \frac{[x \dot{-} r^v]}{r^v})$ will be $= [x \dot{-} v] \times \text{L} : r + v \times \frac{[x \dot{-} r]}{r}$: Consequently $[x \dot{-} r^v]$ will then be $= r^v \times [x \dot{-} v] \times \text{L} : r + r^{v-1} v [x \dot{-} r]$.

By proceeding much in the same manner, may the values of $[x \dot{-} \overline{r}|^{w}]$ and $[x \dot{-} \overline{r}|^{v^w}]$, &c. be found in terms of r, v, w, x, $[x \dot{-} r]$, $[x \dot{-} v]$, $[x \dot{-} w]$, &c.

EXAM-

EXAMPLE I. N and n being invariable, $[x \perp N^{nx}]$ is $= nN^{nx} \times L : N$.

EXAMPLE II. If a, b, and n be invariable; $[x \perp \overline{a+bx}|^{nx}]$ will be $= n \times \overline{a+bx}|^{nx} \times L : \overline{a+bx} + bnx \times \overline{a+bx}|^{nx-1}$.

EXAMPLE III. If (A, N, and a, b, c, &c. being invariable)

$$AN^{x} \text{ be } = \overline{y+a} \cdot \overline{y+b} \cdot \overline{y+c}\,(n).$$

we, by residual division, shall have

$AN^{x} \times [y \perp x] \times L : N = \overline{y+b} \cdot \overline{y+c} \cdot \overline{y+d}\,(n-1) + \overline{y+a} \cdot \overline{y+c} \cdot \overline{y+d}\,(n-1) + \overline{y+a} \cdot \overline{y+b} \cdot \overline{y+d}\,(n-1) + (n)$.

II.

By the binomial theorem, investigated in Article 1, we have $\overline{1+z}|^{v} = 1 + vz + \frac{v \cdot v-1}{2} \cdot z^{2} + \frac{v \cdot v-1 \cdot v-2}{2 \cdot 3} \cdot z^{3}$ &c. from whence, by residual division, ($v - \underset{\prime}{v}$ being the divisor, and the value of z independent of v,) we get

$$\overline{1+z}|^{v} \times L : \overline{1+z} = z + \left.\begin{matrix} v-1 \\ v \end{matrix}\right\} \frac{z^{2}}{2} + \left.\begin{matrix} v-1 \cdot v-2 \\ v \cdot v-2 \\ v \cdot v-1 \end{matrix}\right\} \frac{z^{3}}{2 \cdot 3} \text{ \&c.}$$

Hence, supposing $v = 0$, or any positive integer, we find

$$L : \overline{1+z} = \frac{P}{\overline{1+z}|^{v}} + \frac{1 \cdot 1 \cdot 2 \cdot 3\,(v+1)}{\overline{1+z}|^{v}} \times \overline{\frac{z^{v+1}}{1 \cdot 2 \cdot 3\,(v+1)} - \frac{z^{v+2}}{2 \cdot 3 \cdot 4\,(v+1)} + \frac{z^{v+3}}{3 \cdot 4 \cdot 5\,(v+1)} - \frac{z^{v+4}}{4 \cdot 5 \cdot 6\,(v+1)} + \text{\&c.}}$$

P being put for $z + \left.\begin{matrix} v-1 \\ v \end{matrix}\right\} \frac{z^{2}}{2} + \left.\begin{matrix} v-1 \cdot v-2 \\ v \cdot v-2 \\ v \cdot v-1 \end{matrix}\right\} \frac{z^{3}}{2 \cdot 3}\,(v)$.

12. Again,

12.

Again, by the binomial theorem we have $\overline{z+d}|^{x+1} = z^{x+1} + \overline{x+1}\,.\,dz^x + \frac{\overline{x+1}\,.\,x}{2}d^2z^{x-1} + \frac{\overline{x+1}\,.\,x\,.\,\overline{x-1}}{2\,.\,3}d^3z^{x-2}$ &c.

Therefore

$$z^x \text{ is } = \frac{\overline{z+d}|^{x+1} - z^{x+1}}{d\,.\,\overline{x+1}} - \frac{x}{2}dz^{x-1} - \frac{x\,.\,\overline{x-1}}{2\,.\,3}d^2z^{x-2} \text{ \&c.}$$

Hence, expounding z by a, $a+d$, $a+2d$, &c. successively, we have

$$a^x = \frac{\overline{a+d}|^{x+1} - a^{x+1}}{d\,.\,\overline{x+1}} - \frac{x}{2}d\,.\,a^{x-1} - \frac{x\,.\,\overline{x-1}}{2\,.\,3}d^2\,.\,a^{x-2} \text{ \&c.}$$

$$\overline{a+d}|^x = \frac{\overline{a+2d}|^{x+1} - \overline{a+d}|^{x+1}}{d\,.\,\overline{x+1}} - \frac{x}{2}d\,.\,\overline{a+d}|^{x-1} - \frac{x\,.\,\overline{x-1}}{2\,.\,3}d^2\,.\,\overline{a+d}|^{x-2} \text{ \&c.}$$

$$\overline{a+2d}|^x = \frac{\overline{a+3d}|^{x+1} - \overline{a+2d}|^{x+1}}{d\,.\,\overline{x+1}} - \frac{x}{2}d\,.\,\overline{a+2d}|^{x-1} - \frac{x\,.\,\overline{x-1}}{2\,.\,3}d^2\,.\,\overline{a+2d}|^{x-2} \text{ \&c.}$$

&c. &c. &c.

Consequently, we get, by addition

$$\underline{S}^x = \frac{\overline{a+nd}|^{x+1} - a^{x+1}}{d\,.\,\overline{x+1}} - \frac{x}{2}d\underline{S}^{x-1} - \frac{x\,.\,\overline{x-1}}{2\,.\,3}d^2\underline{S}^{x-2} \text{ \&c.}$$

$\underline{S}^x$ being wrote for $a^x + \overline{a+d}|^x + \overline{a+2d}|^x + \overline{a+3d}|^x$ (n),

$\underline{S}^{x-1}$ for $a^{x-1} + \overline{a+d}|^{x-1} + \overline{a+2d}|^{x-1} + \overline{a+3d}|^{x-1}$ (n),

&c. &c.

From

From whence it follows, that

$$\underset{\div}{S}^{x-1} \text{ is } = \frac{\overline{a+nd}|^{x} - a^{x}}{dx} - \frac{x-1}{2} d\underset{\div}{S}^{x-2} - \frac{x-1 \cdot x-2}{2 \cdot 3} d^2 \underset{\div}{S}^{x-3} \text{ \&c.}$$

$$\underset{\div}{S}^{x-2} = \frac{\overline{a+nd}|^{x-1} - a^{x-1}}{d \cdot \overline{x-1}} - \frac{x-2}{2} d\underset{\div}{S}^{x-3} - \frac{x-2 \cdot x-3}{2 \cdot 3} d^2 \underset{\div}{S}^{x-4} \text{ \&c.}$$

&c. &c. &c. &c.

Now theſe values of $\underset{\div}{S}^{x-1}$, $\underset{\div}{S}^{x-2}$, $\underset{\div}{S}^{x-3}$, &c. being ſubſtituted ſucceſſively, for their reſpective equals, in the value of $\underset{\div}{S}^{x}$, we find

$$\underset{\div}{S}^{x} = a^{x} + \overline{a+d}|^{x} + \overline{a+2d}|^{x} + \overline{a+3d}|^{x} \, (n) =$$

$$\frac{1}{d \cdot \overline{x+1}} \begin{cases} \overline{a+nd}|^{x+1} \\ -a^{x+1} \end{cases} - \frac{1}{2} \begin{cases} \overline{a+nd}|^{x} \\ -a^{x} \end{cases} + \frac{x\overset{\prime}{A}d}{2} \begin{cases} \overline{a+nd}|^{x-1} \\ -a^{x-1} \end{cases} +$$

$$\frac{x \cdot x-1 \cdot x-2 \cdot \overset{\prime\prime}{A}d^3}{2 \cdot 3 \cdot 4} \begin{cases} \overline{a+nd}|^{x-3} \\ -a^{x-3} \end{cases} + \frac{x \cdot x-1 \cdot x-2 \cdot x-3 \cdot x-4 \cdot \overset{\prime\prime\prime}{A}d^5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \begin{cases} \overline{a+nd}|^{x-5} \\ -a^{x-5} \end{cases} \text{ \&c.}$$

$\overset{\prime}{A}$ being $= \frac{1}{2 \cdot 3} = \frac{1}{6}$,

$$\overset{\prime\prime}{A} = \frac{3}{2 \cdot 5} - \frac{4}{2}\overset{\prime}{A} = -\frac{1}{30},$$

$$\overset{\prime\prime\prime}{A} = \frac{5}{2 \cdot 7} - \frac{6}{2}\overset{\prime}{A} - \frac{6 \cdot 5 \cdot 4}{2 \cdot 3 \cdot 4}\overset{\prime\prime}{A} = \frac{1}{42},$$

$$\overset{\prime\prime\prime\prime}{A} = \frac{7}{2 \cdot 9} - \frac{8}{2}\overset{\prime}{A} - \frac{8 \cdot 7 \cdot 6}{2 \cdot 3 \cdot 4}\overset{\prime\prime}{A} - \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}\overset{\prime\prime\prime}{A} = -\frac{1}{30},$$

$$\overset{v}{A} = \frac{9}{2 \cdot 11} - \frac{10}{2}\overset{\prime}{A} - \frac{10 \cdot 9 \cdot 8}{2 \cdot 3 \cdot 4}\overset{\prime\prime}{A} - \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}\overset{\prime\prime\prime}{A} - \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}\overset{\prime\prime\prime\prime}{A} = \frac{5}{66},$$

&c. &c. &c.

SCHOLIUM. When x is $= -1$, the numerator and denominator of the fraction $\frac{\overline{a+nd}|^{x+1} - a^{x+1}}{d \cdot \overline{x+1}}$ $\left(= \frac{1}{d \cdot \overline{x+1}} \begin{cases} \overline{a+nd}|^{x+1} \\ -a^{x+1} \end{cases}\right)$

both

both vanish.—In that case $\frac{1}{d} \times \mathrm{L} : \frac{a+nd}{a}$ is the value of that fraction, as appears by what is already taught.

13.

Writing $2n$ instead of n, we have, by the preceding article,

$$a^x = \overline{a+d}|^x + \overline{a+2d}|^x + \overline{a+3d}|^x (2n) = \frac{1}{d.\overline{x+1}}\left\{\begin{array}{l}\overline{a+2nd}|^{x+1}\\ -a^{x+1}\end{array}\right.$$

$$-\frac{1}{2}\left\{\begin{array}{l}\overline{a+2nd}|^{x}\\ -a^{x}\end{array}\right. + \frac{x\overset{\prime}{A}d}{2}\left\{\begin{array}{l}\overline{a+2nd}|^{x-1}\\ -a^{x-1}\end{array}\right. + \frac{x.x-1.x-2\overset{\prime\prime}{A}d^3}{2\cdot3\cdot4}\left\{\begin{array}{l}\overline{a+2nd}|^{x-3}\\ -a^{x-3}\end{array}\right.$$

&c. and, writing $2d$ instead of d, we have, by the same article,

$$a^x + \overline{a+2d}|^x + \overline{a+4d}|^x + \overline{a+6d}|^x (n) = \frac{1}{2d.\overline{x+1}}\left\{\begin{array}{l}\overline{a+2nd}|^{x+1}\\ -a^{x+1}\end{array}\right.$$

$$-\frac{1}{2}\left\{\begin{array}{l}\overline{a+2nd}|^{x}\\ -a^{x}\end{array}\right. + \frac{2x\overset{\prime}{A}d}{2}\left\{\begin{array}{l}\overline{a+2nd}|^{x-1}\\ -a^{x-1}\end{array}\right. + \frac{2^3.x.x-1.x-2.\overset{\prime\prime}{A}d^3}{2\cdot3\cdot4}\left\{\begin{array}{l}\overline{a+2nd}|^{x-3}\\ -a^{x-3}\end{array}\right.$$

&c.

It appears, therefore, by subtraction, that

$$\overline{a+d}|^x + \overline{a+3d}|^x + \overline{a+5d}|^x (n) \text{ is} = \frac{1}{2d.\overline{x+1}}\left\{\begin{array}{l}\overline{a+2nd}|^{x+1}\\ -a^{x+1}\end{array}\right.$$

$$-\frac{x\overset{\prime}{A}d}{2}\left\{\begin{array}{l}\overline{a+2nd}|^{x-1}\\ -a^{x-1}\end{array}\right. - \frac{\overline{2^3-1}.x.x-1.x.-2.\overset{\prime\prime}{A}d^3}{2.3.4}\left\{\begin{array}{l}\overline{a+2nd}|^{x-3}\\ -a^{x-3}\end{array}\right. -$$

$$\frac{\overline{2^5-1}.x.x-1.x-2.x-3.x-4.\overset{\prime\prime\prime}{A}d^5}{2.3.4.5.6}\left\{\begin{array}{l}\overline{a+2nd}|^{x-5}\\ -a^{x-5}\end{array}\right. \text{\&c.}$$

SCHOL. When x is $= -1$, the value of $\frac{1}{2d.\overline{x+1}}\left\{\begin{array}{l}\overline{a+2nd}|^{x+1}\\ -a^{x+1}\end{array}\right.$ is $= \frac{1}{2d} \times \mathrm{L} : \frac{a+2nd}{a}$, by what is done above.

14. Taking

14.

Taking the laſt equation from that which immediately precedes it, we have

$$a^x - \overline{a+d}|^x + \overline{a+2d}|^x - \overline{a+3d}|^x + \overline{a+4d}|^x - \overline{a+5d}|^x (2n) =$$

$$-\frac{1}{2}\left\{\begin{array}{l}\overline{a+2nd}|^x \\ -a^x\end{array}\right. + \frac{\overline{2^2-1}\,.\,x\overset{\prime}{A}d}{2}\left\{\begin{array}{l}\overline{a+2nd}|^{x-1} \\ -a^{x-1}\end{array}\right. + \frac{\overline{2^4-1}\,.\,x\,.\,x-1\,.\,x-2\,.\,\overset{\prime\prime}{A}d^3}{2\,.\,3\,.\,4}$$

$$\left\{\begin{array}{l}\overline{a+2nd}|^{x-3} \\ -a^{x-3}\end{array}\right. + \frac{\overline{2^6-1}\,.\,x\,.\,x-1\,.\,x-2\,.\,x-3\,.\,x-4\,.\,\overset{\prime\prime\prime}{A}d^3}{2\,.\,3\,.\,4\,.\,5\,.\,6}\left\{\begin{array}{l}\overline{a+2nd}|^{x-5} \\ -a^{x-5}\end{array}\right. \&c.$$

n being any poſitive integer.

15.

By reſidual diviſion, ($x - x_{\prime}$ being the diviſor, and the values of a and d independent of x,) we get, from Art. 12.

$$a^x \times L : a + \overline{a+d}|^x \times L : \overline{a+d} + \overline{a+2d}|^x \times L : \overline{a+2d} +$$

$$\overline{a+3d}|^x \times L : \overline{a+3d}\ (n)$$

$$=\left\{\begin{array}{l}
\frac{1}{d\,.\,\overline{x+1}}\left\{\begin{array}{l}\overline{a+nd}|^{x+1} \times L : \overline{a+nd} \\ -a^{x+1} \times L : a\end{array}\right. - \frac{1}{2}\left\{\begin{array}{l}\overline{a+nd}|^x \times L : \overline{a+nd} \\ -a^x \times L : a\end{array}\right. \\
\qquad + \frac{x\overset{\prime}{A}d}{2}\left\{\begin{array}{l}\overline{a+nd}|^{x-1} \times L : \overline{a+nd} \\ -a^{x-1} \times L : a\end{array}\right. \&c. \\
\frac{-1}{d\,.\,\overline{x+1}|^2}\left\{\begin{array}{l}\overline{a+nd}|^{x+1} \\ -a^{x+1}\end{array}\right. + \frac{\overset{\prime}{A}d}{2}\left\{\begin{array}{l}\overline{a+nd}|^{x-1} \\ -a^{x-1}\end{array}\right. + \left.\begin{array}{r}x-1\,.\,x-2 \\ x\,.\,x-2 \\ x\,.\,x-1\end{array}\right\}\times \\
\qquad \frac{\overset{\prime\prime}{A}d^3}{2\,.\,3\,.\,4}\left\{\begin{array}{l}\overline{a+nd}|^{x-3} \\ -a^{x-3}\end{array}\right. \&c.
\end{array}\right.$$

Corollary I. Suppoſing $x = 0$, we find

$$L:a + L:\overline{a+d} + L:\overline{a+2d} + L:\overline{a+3d}\,(n)$$

$$= \begin{cases} \overline{\frac{a}{d} + n - \frac{1}{2}} \times L:\overline{a+nd} - \overline{\frac{a}{d} - \frac{1}{2}} \times L:a, - n + \\ \frac{\acute{A}d}{1.2}\begin{cases}\overline{a+nd}|^{-1} \\ -a^{-1}\end{cases} + \frac{\acute{\acute{A}}d^3}{3.4}\begin{cases}\overline{a+nd}|^{-3} \\ -a^{-3}\end{cases} + \frac{\acute{\acute{\acute{A}}}d^5}{5.6}\begin{cases}\overline{a+nd}|^{-5} \\ -a^{-5}\end{cases} \text{\&c.}\end{cases}$$

Corollary II. Taking, in the preceding Corollary, a, d, and n each equal to 1, we have

$$\frac{3}{2} \times L:2 = 1 + \frac{\acute{A}}{2.1.2} + \frac{\overline{2^3-1}.\acute{\acute{A}}}{2^3.3.4} + \frac{\overline{2^5-1}.\acute{\acute{\acute{A}}}}{2^5.5.6} \text{ \&c.}$$

16.

As the principal theorem in the laſt article was deduced from Art. 12. ſo from Art. 13. we deduce the following theorem, viz.

$$\overline{a+d}|^x \times L:\overline{a+d} + \overline{a+3d}|^x \times L:\overline{a+3d} + \overline{a+5d}|^x \times L:\overline{a+5d}\,(n)$$

$$= \begin{cases} \frac{1}{2d.\overline{x+1}}\begin{cases}\overline{a+2nd}|^{x+1} \times L:\overline{a+2nd} \\ -a^{x+1} \times L:a\end{cases} - \frac{x\acute{A}d}{2} \times \\ \qquad \begin{cases}\overline{a+2nd}|^{x-1} \times L:\overline{a+2nd} \\ -a^{x-1} \times L:a\end{cases} \text{\&c.} \\ \frac{-1}{2d.\overline{x+1}|^2}\begin{cases}\overline{a+2nd}|^{x+1} \\ -a^{x+1}\end{cases} - \frac{\acute{A}d}{2}\begin{cases}\overline{a+2nd}|^{x-1} \\ -a^{x-1}\end{cases} - \begin{matrix} x-1.x-2 \\ x.x-2 \\ x.x-1\end{matrix}\Bigg\} \times \\ \qquad \frac{\overline{2^3-1}.\acute{\acute{A}}d^3}{2.3.4}\begin{cases}\overline{a+2nd}|^{x-3} \\ -a^{x-3}\end{cases} \text{\&c.}\end{cases}$$

Corollary I. Hence, ſuppoſing $x = 0$, we find

$$L:\overline{a+d} + L:\overline{a+3d} + L:\overline{a+5d} + L:\overline{a+7d}\,(n)$$

$=$

$$=\begin{cases}\overline{\frac{a}{2d}+n}\times L:\overline{a+2nd}-\frac{a}{2d}\times L:a \\ -n-\frac{\acute{A}d}{1.2}\begin{Bmatrix}\overline{a+2nd}\vert^{-1}\\ -a^{-1}\end{Bmatrix}-\frac{\overline{2^3-1}\,.\,\overset{\prime\prime}{A}d^3}{3.4}\begin{Bmatrix}\overline{a+2nd}\vert^{-3}\\ -a^{-3}\end{Bmatrix} \\ \qquad -\frac{\overline{2^5-1}\,.\,\overset{\prime\prime\prime}{A}d^5}{5.6}\begin{Bmatrix}\overline{a+2nd}\vert^{-5}\\ -a^{-5}\end{Bmatrix}\ \&c.\end{cases}$$

COROLLARY II. Taking, in the preceding Corollary, $a=2$, $d=1$, and writing $n-1$ inſtead of n, we have

$$L:1+L:3+L:5+L:7\,(n)=L:1\times3\times5\times7\,(n)$$

$$=L:2^{n-1}n^n+\begin{cases}1+\frac{\acute{A}}{2.1.2}+\frac{\overline{2^3-1}\,.\,\overset{\prime\prime}{A}}{2^3.3.4}+\frac{\overline{2^5-1}\,.\,\overset{\prime\prime\prime}{A}}{2^5.5.6}\ \&c. \\ -n-\frac{\acute{A}}{2.1.2.n}-\frac{\overline{2^3-1}\,.\,\overset{\prime\prime}{A}}{2^3.3.4.n^3}-\frac{\overline{2^5-1}\,.\,\overset{\prime\prime\prime}{A}}{2^5.5.6.n^5}\ \&c.\end{cases}$$

$=n\times L:n+\overline{n+\frac{1}{2}}\times L:2,\ -n-\frac{\acute{A}}{2.1.2.n}-\frac{\overline{2^3-1}\,.\,\overset{\prime\prime}{A}}{2^3.3.4.n^3}-\frac{\overline{2^5-1}\,.\,\overset{\prime\prime\prime}{A}}{2^5.5.6.n^5}$ &c. $\frac{3}{2}\times L:2$ being $=1+\frac{\acute{A}}{2.1.2}+\frac{\overline{2^3-1}\,.\,\overset{\prime\prime}{A}}{2^3.3.4}+\frac{\overline{2^5-1}\,.\,\overset{\prime\prime\prime}{A}}{2^5.5.6}$ &c. by Cor. 2. of the laſt Article.

Other theorems may, in like manner, be derived from Art. 14. which we may take notice of in an Appendix to this Treatiſe; and perhaps add ſome farther improvements on the ſubject of the laſt five articles, which ſome time ago engaged the attention of Mr. DE MOIVRE, Mr. STIRLING, and other eminent mathematicians.

THE

THE

RESIDUAL ANALYSIS.

CHAP. IV.

Of the Properties *of certain* ALGEBRAIC EXPRESSIONS.

THE articles in this chapter will be found of considerable uſe, in geometrical and phyſical enquiries; and, to the end that we may proceed with as much perſpicuity as poſſible, it is thought proper to inſert them here, previous to ſuch enquiries.

I.

Suppoſe E *to be an algebraic expreſſion compoſed of* x *and other quantities; and ſuppoſe, that, how near ſoever* x *be taken to ſome certain quantity* g, E *is poſitive when* x *is leſs than* g, *and negative when* x *is greater than* g; *or poſitive when* x *is greater than* g, *and negative when* x *is leſs than* g: *then ſhall* E, *or its reciprocal, be* = o *when* x *is* = g.

For it is well known, that o is the only limit between poſitive and negative; and it is therefore plain, that either the value of E muſt continually approach to that limit, or increaſe or decreaſe without limit, upon taking x nearer and nearer to g. If the value of E approaches to o when x is taken nearer and nearer

nearer to g, it is evident that E will be $= 0$ when x is $= g$: If the value of E increafes or decreafes without limit when x is taken nearer and nearer to g, the value of $\frac{1}{E}$ will then continually approach to 0; therefore, $\frac{1}{E}$ being pofitive or negative according as E is pofitive or negative, it is evident that, in fuch cafe, $\frac{1}{E}$ will be $= 0$ when x is $= g$. Confequently either E or its reciprocal muft be $= 0$ when x is $= g$.

2.

Q being an algebraic expreffion fo compofed of x *and other quantities, that neither it nor its reciprocal vanifhes when* x *is therein taken equal to* g; *the faid expreffion, when* x *is greater or lefs than* g, *between certain limits, will (fuppofing it not to become imaginary) be pofitive or negative, according as* (q) *the value of* Q, *when* x *is equal to* g, *is pofitive or negative.*

For, let A and B be pofitive quantities; and fuppofe that either Q or $\frac{1}{Q}$ is $= 0$ when x is $= g + A$, and likewife that Q or $\frac{1}{Q}$ is $= 0$ when x is $= g - B$, but that neither Q nor $\frac{1}{Q}$ is $= 0$ when x is taken between the limits $g + A$ and $g - B$: then, it is obvious, that, whilft x is taken between thofe limits, Q (if it does not become imaginary) will be always pofitive, or always negative; and, confequently, pofitive or negative according as q is pofitive or negative.

The fame conclufion follows, if, inftead of Q or $\frac{1}{Q}$ being $= 0$ when x is $= g + A$, or when x is $= g - B$, thofe expreffions then become imaginary, but are real and finite when x is taken between the limits above-mentioned.

3.

Let m be an odd number, or a fraction whofe numerator and denominator are both odd numbers; and P any algebraic expreffion fo compofed of x and other quantities, that its value fhall be real when x is either greater or lefs than g, (between certain

certain limits,) and that neither it nor its reciprocal ſhall vaniſh when x is equal to g; alſo let p be the value of P when x is $= g$.

Then, ſeeing that $\overline{x-g}|^m$ is poſitive or negative, according as x is greater or leſs than g; it is evident, (from what is ſaid in the laſt Article) that, how near ſoever x be taken to g, if p be poſitive, $\overline{x-g}|^m \times P$ will be negative when x is leſs than g, and poſitive when x is greater than g; or, if p be negative, the ſame expreſſion $(\overline{x-g}|^m \times P)$ will be negative when x is greater than g, and poſitive when x is leſs than g.

Corollary. Suppoſe E to be an algebraic expreſſion compoſed of x and other quantities: Then, if E be poſitive when x is greater than g, and negative when x is leſs than g, how near ſoever x be taken to g; it is manifeſt, from what we juſt now obſerved, that

$$\overline{x-g}|^m \times Q \text{ may be aſſumed } = E;$$

m being as above ſpecified, and Q ſome algebraic expreſſion conſiſting of ſuch quantities, that its value ſhall be real when x is either greater or leſs than g, (between certain limits,) and that (q) its value when x is equal to g ſhall be finite and poſitive.

Moreover, if E be poſitive when x is leſs than g, and negative when x is greater than g, how near ſoever x be taken to g; it is likewiſe manifeſt, that, in this caſe alſo,

$$\overline{x-g}|^m \times Q \text{ may be aſſumed } = E;$$

m and Q being as before ſpecified, except that q, inſtead of being poſitive, muſt be negative.

Hence it is evident, that, when x is $= g$, (whether q be poſitive or negative,) E or $\frac{1}{E}$ will be $= 0$, according as m is poſitive or negative; which agrees with what is ſaid in Art. 1.

THE

THE

RESIDUAL ANALYSIS.

CHAP. V.

Of the TANGENTS *of* curve Lines.

I.

THE curve $AqPq$, having its convexity downwards, as in Fig. 1. being referred to the base AB; if the right line $NrPr$ touch the said curve in P; and, brq being parallel to BP, if AB be called x; BP, y; Ab, $x_{,}$; bq, $y_{,}$; and the subtangent BN, s: then will bN be $= s - x + x_{,}$, $br = y - \frac{y}{s} \times \overline{x - x_{,}}$, and the residual $bq - br$ $(= qr) =$ $\frac{y}{s} \times \overline{x - x_{,}} - \overline{y - y_{,}}$. Now, brq being drawn on either side of BP, bq is manifestly greater than br; therefore $\frac{y}{s} \times \overline{x - x_{,}}$ $- \overline{y - y_{,}}$ (the value of $bq - br$) must be always positive, when, x being of any given value, $x_{,}$ is either less or greater than x. Fig. 1.

But if the convexity of the curve be upwards, as in Fig. 2. $\frac{y}{s} \times \overline{x - x_{,}} - \overline{y - y_{,}}$ (the value of $bq - br$) must be always Fig. 2.

negative, when, x being of any given value, $\underset{,}{x}$ is either lefs or greater than x.

Now, fince the expreffion $\frac{y}{s} \times \overline{x - \underset{,}{x}} - \overline{y - \underset{,}{y}}$ muft be always pofitive or always negative, when, x being of any given value, $\underset{,}{x}$ is either lefs or greater than x, the quotient of that expreffion divided by $x - \underset{,}{x}$ (viz. $\frac{y}{s} - [x \mid y]$) will, it is obvious, be pofitive when $\underset{,}{x}$ is lefs than x, and negative when $\underset{,}{x}$ is greater than x; or pofitive when $\underset{,}{x}$ is greater than x, and negative when $\underset{,}{x}$ is lefs than x, how near foever $\underset{,}{x}$ be taken to x. Therefore, by Chap. 4. Art. 1.

$$\frac{y}{s} - [x \,\dot{-}\, y] \text{ will be } = 0 \,*;$$

$$\text{and, confequently, } s = \frac{y}{[x \,\dot{-}\, y]}.$$

Fig. 1. EXAMPLE I. *To draw a tangent to any Parabola, whofe equation is* $ax^{\frac{m}{r}} = y$; m *being fuch, that the convexity of the curve is downwards.*

We have (according to our fcheme) $bq - br$ $(= qr) = \frac{ax^{\frac{m}{r}}}{s} \times \overline{x - \underset{,}{x}} - ax^{\frac{m}{r}} + \underset{,}{a}x^{\frac{m}{r}}$; where s muft be of fuch a value, that the value of the whole expreffion fhall be pofitive, when, x being of any given value, $\underset{,}{x}$ is either greater or lefs than x.

Now if $\frac{ax^{\frac{m}{r}}}{s} \times \overline{x - \underset{,}{x}} - ax^{\frac{m}{r}} + a\underset{,}{x}^{\frac{m}{r}}$ be always pofitive when $\underset{,}{x}$ is either greater or lefs than x, $\dfrac{\frac{ax^{\frac{m}{r}}}{s} \times \overline{x - \underset{,}{x}} - ax^{\frac{m}{r}} + a\underset{,}{x}^{\frac{m}{r}}}{x - \underset{,}{x}}$, or its

* The other equation (viz. $\dfrac{1}{\frac{y}{s} - [x \,\dot{-}\, y]} = 0$) which, if poffible, wou'd follow from the fame article, is, in this cafe, manifeftly impoffible.

Equal

Equal $\frac{ax^{\frac{m}{r}}}{s} - ax^{\frac{m}{r} - 1} \times \dfrac{1 + \frac{x_{,}}{x} + \left(\frac{x_{,}}{x}\right)^{2} \ldots (m)}{1 + \left(\frac{x_{,}}{x}\right)^{\frac{m}{r}} + \left(\frac{x_{,}}{x}\right)^{\frac{m}{r}} \ldots (r)}$ (found by our division taught Chap. 2. Art. 1.) it is plain will be positive when $x_{,}$ is less than x, and negative when $x_{,}$ is greater than x. Therefore, by Chap. 4. Art. 1. the value of this last expression will be $= 0$ when $x_{,}$ is therein taken equal to x: Which value is manifestly $= \frac{ax^{\frac{m}{r}}}{s} - \frac{m}{r} . ax^{\frac{m}{r} - 1}$.

Consequently from the equation $\frac{ax^{\frac{m}{r}}}{s} - \frac{m}{r} . ax^{\frac{m}{r} - 1} = 0$, the subtangent s is found $= \frac{r}{m}x$.

In this example we have given the process at full length, that the Reader may the more clearly understand our doctrine.— In future, our examples will, for the most part, be more concise.

EXAMPLE II. *To draw a tangent to the Circle* AqPq, *whose equation is* $2ax - x^2 = y^2$; *the radius* AC *being* $= a$, AB $= x$, BP $= y$. Fig. 3.

From the equation of the curve, we have, by residual division, $a - x = y \times [x \perp y]$; from whence we get $[x \perp y] = \frac{a - x}{y}$. Therefore $\frac{y}{[x \perp y]}$, the value of s or NB, is $= \frac{y^2}{a - x} = \frac{2ax - x^2}{a - x}$.

If BC be called x, and BP as before, the equation of the curve will be $a^2 - x^2 = y^2$; and the subtangent NB will be $= \frac{a^2 - x^2}{x}$.

EXAMPLE III. *To draw a tangent to the Ellipsis* APD, *whose equation is* $b^2 \times \overline{2ax - x^2} = a^2y^2$; *the semi-transverse axis* AC *being* $= a$, *the semi-conjugate* CD $= b$, AB $= x$, BP $= y$. Fig. 4.

From the equation of the curve, we get, by reſidual diviſion, $ab^2 - b^2x = a^2y[x \perp y]$: hence $[x \perp y]$ is found $= \frac{b^2}{a^2} \times \frac{a-x}{y}$.

Therefore $\frac{y}{[x \perp y]}$, the value of the ſubtangent NB, is $= \frac{a^2y^2}{b^2 \times \overline{a-x}} = \frac{2ax - x^2}{a-x}$.

If BC be called x, and BP as before; the equation of the curve will be $b^2 \times \overline{a^2 - x^2} = a^2y^2$; and NB will be $= \frac{a^2 - x^2}{x}$.

If, Pd being parallel to AC, Cd be called x; and dP, y: the equation of the curve will be $a^2 \times \overline{b^2 - x^2} = b^2y^2$; and the ſubtangent $dn = \frac{b^2 - x^2}{x}$.

Fig. 5. EXAMPLE IV. *Let it be propoſed to draw a tangent to the Hyperbola* AP, *whoſe equation is* $b^2 \times \overline{2ax + x^2} = a^2y^2$; *the ſemi-tranverſe axis* AC *being* $= a$, *the ſemi-conjugate* CD $= b$, AB $= x$, BP $= y$.

From the equation of the curve we have $ab^2 + b^2x = a^2y[x \perp y]$: hence we find $[x \perp y] = \frac{b^2}{a^2} \times \frac{a+x}{y}$. Therefore $\frac{y}{[x \perp y]}$, the value of the ſubtangent NB, is $= \frac{a^2y^2}{b^2 \times \overline{a+x}} = \frac{2ax + x^2}{a+x}$.

If BC be called x, and BP as before; the equation of the curve will be $b^2 \times \overline{x^2 - a^2} = a^2y^2$; and NB will be $= \frac{x^2 - a^2}{x}$.

If, Pd being parallel to AC, Cd be called x; and dP, y; the equation of the curve will be $a^2 \times \overline{b^2 + x^2} = b^2y^2$; and the ſubtangent $dn = \frac{b^2 + x^2}{x}$.

Fig. 6. Suppoſe CE, Ce to be the aſymptotes of the two branches AP, AQ, of the Hyperbola PAQ: then, BP being parallel to Ce, if CB, BP be called x and y reſpectively; the equation of the curve will be $xy = p^2$, p being put for $\frac{\sqrt{a^2 + b^2}}{2}$. Therefore,

$y +$

$y + x[x \perp y]$ being $= 0$, $[x \perp y]$ is $= -\frac{y}{x}$, and $\frac{y}{[x \perp y]}$ ($=$ NB) $= -x$; which being negative, indicates, that N is on the contrary fide of BP, from C.

EXAMPLE V. *To draw a tangent to the Ciffoid* AP, *whofe equation is* $ay^2 - xy^2 = x^3$. Fig. 7.

From the faid equation we have $2ay[x \perp y] - y^2 - 2xy[x \perp y] = 3x^2$; and from hence we get $[x \perp y] = \frac{3x^2 + y^2}{2ay - 2xy}$: Which laft expreffion being fubftituted for $[x \perp y]$ in the equation $s = \frac{y}{[x \perp y]}$, we find $s = \frac{2ay^2 - 2xy^2}{3x^2 + y^2}$.

EXAMPLE VI. *To draw a tangent to the exponential curve, whofe equation is* $a^x = y$.

By refidual divifion (fee Chap. 3. Art. 9.) we get $a^x \times L:a = [x \perp y]$: Therefore $s \left(= \frac{y}{[x \perp y]}\right)$ is $= \frac{1}{L:a}$, an invariable quantity.

EXAMPLE VII. *To draw a tangent to the exponential curve, whofe equation is* $x^x = y$.

From the faid equation we find, by refidual divifion, $x^x \times L:x + x^x = [x \perp y]$: Confequently $s \left(= \frac{y}{[x \perp y]}\right)$ is $= \frac{1}{1 + L:x}$.

After the fame manner may we draw tangents to any other geometrical, or exponential curve, referred to an axis: but to perform the like by our Analyfis, when the curve is a tranfcendental one, or a fpiral, it will be neceffary to underftand what will be explained in the fubfequent part of this chapter.

2.

A line has its concavity turned one way, when the right lines that join any two of its points either fall upon the line itfelf, or on one fide of it, none falling on the oppofite fide.—Here we comprehend, not only curves, but likewife fuch lines as have rectilineal parts.

3. When

3.

When two lines, having their concavity turned one and the ſame way, have the ſame terms, and one wholly includes the other, the perimeter of that which includes is greater than the perimeter of that which is included *.

4.

Fig. 8. When a curve hqP is convex towards the baſe, and the angle BPN, made by the ordinate BP and tangent PN, is acute; the ordinate bq being drawn, interſecting NP in r, Pr will be greater or leſs than the curve Pq, according as Ab is leſs or greater than AB. For, drawing a tangent to the point q, (between h and P,) interſecting Pr in v; Pvq, by the preceding Article, will be greater than the curve Pq. But rv, which ſubtends an obtuſe angle in the triangle qrv, is greater than qv, which ſubtends an acute angle in the ſame triangle: therefore Pvr (i. e. Pr) will be greater than Pvq; and conſequently Pr ſtill greater than the curve Pq. Moreover, q being on the other ſide of P; Pr, which ſubtends an acute angle in the triangle Pqr, is leſs than the chord Pq, which ſubtends an obtuſe angle in the ſame triangle: therefore, the chord being leſs than the arc it ſubtends, Pr will be ſtill leſs than the curve Pq.

Fig. 9. When the curve hqP is concave towards the baſe, and the angle BPN is acute; the ordinate bq being drawn, interſecting (as before) the tangent NP in r, it is evident, from what is ſaid above, that Pr will be leſs or greater than the curve Pq, according as Ab is leſs or greater than AB.

5.

Fig. 8. 9. Let AB be called x; BP, y; Ab, $x_{,}$; bq, (parallel to BP,) $y_{,}$; the ſubtangent BN, s; the tangent PN, t; and the parts hP, hq of the curve, z and $z_{,}$ reſpectively: then will Pr be $= \frac{t}{s} \times \overline{x - x_{,}}$ or $\frac{t}{s} \times \overline{x_{,} - x}$, and the arc P$q = z - z_{,}$ or $z_{,} - z$, according as $x_{,}$

* The laſt two articles are from ARCHIMEDES *de ſphæra et cylindro.*

is

is leſs or greater than x. Therefore, by the laſt Article, if the convexity of the curve be downwards, as in Fig. 8. $\frac{t}{s} \times \overline{x - x_{\prime}}$ will be greater than $z - z_{\prime}$, when $x_{\prime}$ is leſs than x; and $\frac{t}{s} \times \overline{x_{\prime} - x}$ will be leſs than $z_{\prime} - z$, when $x_{\prime}$ is greater than x. Hence it evidently follows, that, the curve being convex towards the baſe, $\frac{t}{s} \times \overline{x - x_{\prime}}$ will be always greater than $z - z_{\prime}$, when $x_{\prime}$ is taken either greater or leſs than x: conſequently the expreſſion $\frac{t}{s} \times \overline{x - x_{\prime}} - \overline{z - z_{\prime}}$ will be always poſitive when $x_{\prime}$ is ſo taken.

Moreover, if the convexity of the curve be upwards, as in Fig. 9. $\frac{t}{s} \times \overline{x - x_{\prime}}$ being leſs than $z - z_{\prime}$ when $x_{\prime}$ is leſs than x, and $\frac{t}{s} \times \overline{x_{\prime} - x}$ greater than $z_{\prime} - z$ when $x_{\prime}$ is greater than x, $\frac{t}{s} \times \overline{x - x_{\prime}}$ will be always leſs than $z - z_{\prime}$ when $x_{\prime}$ is taken either greater or leſs than x: conſequently, in this caſe, the expreſſion $\frac{t}{s} \times \overline{x - x_{\prime}} - \overline{z - z_{\prime}}$ will be always negative when $x_{\prime}$ is ſo taken.

Now, ſince the expreſſion $\frac{t}{s} \times \overline{x - x_{\prime}} - \overline{z - z_{\prime}}$ muſt be always poſitive or always negative, when, x being of any given value, $x_{\prime}$ is either leſs or greater than x; the quotient of that expreſſion divided by $x - x_{\prime}$ (viz. $\frac{t}{s} - [x \mid z]$) will, it is evident, be poſitive when $x_{\prime}$ is leſs than x, and negative when $x_{\prime}$ is greater than x; or poſitive when $x_{\prime}$ is greater than x, and negative when $x_{\prime}$ is leſs than x, how near ſoever $x_{\prime}$ be taken to x. Therefore, by Chap. 4. Art. 1.

$\frac{t}{s}$ —

$$\frac{t}{s} - [x \mathbin{\dot{-}} z] \text{ will be} = 0\,^*:$$

$$\text{and, consequently, } [x \mathbin{\dot{-}} z] = \frac{t}{s} = \sqrt{1 + [x \mathbin{\dot{-}} y]^2};$$

s being $= \frac{y}{[x \mathbin{\dot{-}} y]}$, and $t = \frac{y\sqrt{1+[x \mathbin{\dot{-}} y]^2}}{[x \mathbin{\dot{-}} y]}$, by what is ſaid in Article 1.

The ſame concluſion, it is obvious, will likewiſe hold true, when the ordinates decreaſe from h towards P.

Corollary I. Hence it is manifeſt, that the ſubtangent, tangent, and ordinate, are to each other as 1, $[x \mathbin{\dot{-}} z]$, and $[x \mathbin{\dot{-}} y]$ reſpectively.

Corollary II. Suppoſe hP to be an arc of a circle: then, if x be the verſed ſine, y the ſine, and a the radius thereof; it is well known, the ſubtangent will be to the tangent, as y to a; and therefore, by the preceding corollary, $y : a :: 1 : [x \mathbin{\dot{-}} z]$. Conſequently $[x \mathbin{\dot{-}} z]$ is then $= \frac{a}{y} = \frac{a}{\sqrt{2ax - x^2}}$.

Moreover, $[x \mathbin{\dot{-}} z]$ being $= \frac{[y \mathbin{\dot{-}} z]}{[y \mathbin{\dot{-}} x]}$ (by Chap. 2. Art. 8.), $\frac{[y \mathbin{\dot{-}} z]}{[y \mathbin{\dot{-}} x]}$ is $= \frac{a}{y}$, and $[y \mathbin{\dot{-}} z] = \frac{a[y \mathbin{\dot{-}} x]}{y}$. But, y being $= \sqrt{2ax - x^2}$, we, by reſidual diviſion, have $1 = \frac{a - x}{\sqrt{2ax - x^2}} \times [y \mathbin{\dot{-}} x]$; and, from hence, $[y \mathbin{\dot{-}} x] = \frac{\sqrt{2ax - x^2}}{a - x} = \frac{y}{\sqrt{a^2 - y^2}}$. Therefore $[y \mathbin{\dot{-}} z]$ is $= \frac{a}{\sqrt{a^2 - y^2}}$.

Let u be the coſine of z; then will $a^2 - y^2$ be $= u^2$; and, conſequently, $[u \mathbin{\dot{-}} y] = \frac{-u}{y}$. Now $[y \mathbin{\dot{-}} z]$ being $= \frac{[u \mathbin{\dot{-}} z]}{[u \mathbin{\dot{-}} y]}$,

* The other equation (viz. $\frac{1}{\frac{t}{s} - [x \mathbin{\dot{-}} z]} = 0$) which, if poſſible, would follow from the ſame article, is, in this caſe, impoſſible.

this

this laſt quantity will be $= \frac{a}{\sqrt{a^2 - y^2}}$; and, therefore, $[u \perp z]$ will be $= \frac{a[u \perp y]}{\sqrt{a^2 - y^2}} = \frac{a[u \perp y]}{u} = -\frac{a}{y} = \frac{-a}{\sqrt{a^2 - u^2}}$.

COROLLARY III. t^2 being $= s^2 + y^2$, we, by reſidual diviſion, have $t[x \perp t] = s[x \perp s] + y[x \perp y]$: Hence $\frac{t}{s} \times [x \perp t] = [x \perp s] + \frac{y}{s} \times [x \perp y]$.

Now AN being called r, s will be $= x - r$; and conſequently $[x \perp s] = 1 - [x \perp r]$: moreover, by what is ſaid above, $\frac{t}{s}$ is $= [x \perp z]$, and $\frac{y}{s} = [x \perp y]$. Therefore $[x \perp z] \times [x \perp t]$ is $= 1 - [x \perp r] + [x \perp y]^2 = [x \perp z]^2 - [x \perp r]$ ($1 + [x \perp y]^2$ being $= [x \perp z]^2$); from whence it appears, that $[x \perp t]$ is $= [x \perp z] - \frac{[x \perp r]}{[x \perp z]}$. Fig. 8.

6.

By means of the concluſions deduced in the preceding article, we are now enabled to apply the theorem we inveſtigated in Art. 1. to tranſcendental curves referred to an axis.

EXAMPLE I. *To draw a tangent to the Cycloid* AP; *whoſe nature is ſuch, that, the ſemicircle* ApD *being deſcribed, and the ordinate* BpP *being drawn perpendicular to the diameter* AD, BP *is* $=$ Bp + (Arc) Ap. Fig. 10.

Let AD be called $2a$; AB, x; BP, y; Bp, u; and the arc Ap, w. Then, y being $= u + w$, $[x \perp y]$ will be $= [x \perp u] + [x \perp w] = [x \perp u] + \frac{a}{u}$. But, by the nature of the circle, u^2 is $= 2ax - x^2$; from whence $[x \perp u]$ is found $= \frac{a - x}{u}$. Therefore, by ſubſtitution, it appears that $[x \perp y]$ is $= \frac{2a - x}{u}$. Conſequently $\frac{y}{[x \perp y]}$, the value of the ſubtangent NB, is $= \frac{uy}{2a - x} = \frac{xy}{u}$.

It is observable, that $\frac{xy}{u}$ ($=$ NB) is to y ($=$ BP), as x ($=$ AB) to u ($=$ Bp): Therefore the tangent NP is parallel to the chord Ap.

Fig. 11. EXAMPLE II. *To draw a tangent to the Quadratrix* APD.

DE being one fourth of the periphery of a circle described about the center C, draw the radius CPG, which suppose equal to a: call AC, b; AB, x; BP, y; EF, v; FG, (parallel to BP,) u; and the arc EG, w. Then, by the nature of the quadratrix, y will be $= \frac{bw}{a}$; from whence we have $[x \,\dot{-}\, y] = \frac{b}{a} \times [x \,\dot{-}\, w] = \frac{b}{a} \times [v \,\dot{-}\, w] \times [x \,\dot{-}\, v] = \frac{b\,[x \,\dot{-}\, v]}{u}$, $\frac{[x \,\dot{-}\, w]}{[x \,\dot{-}\, v]}$ being $= [v \,\dot{-}\, w]$ by Chap. 2. Art. 8. and $[v \,\dot{-}\, w] = \frac{a}{u}$ by the last article. Moreover, by the nature of the circle, u^2 is $= 2av - v^2$: from whence, by residual division, we get $u\,[x \,\dot{-}\, u] = \overline{a - v} \times [x \,\dot{-}\, v]$; and hence $\frac{[x \,\dot{-}\, u]}{a - v} = \frac{[x \,\dot{-}\, v]}{u}$. It follows therefore, that $[x \,\dot{-}\, y]$ is $= \frac{b\,[x \,\dot{-}\, v]}{u} = \frac{b\,[x \,\dot{-}\, u]}{a - v}$; and consequently that $[x \,\dot{-}\, v]$ is $= \frac{u\,[x \,\dot{-}\, y]}{b}$, and $[x \,\dot{-}\, u] = \frac{a - v}{b} \times [x \,\dot{-}\, y]$. Again, by similar triangles, $b - x : y :: a - v : u$; therefore $ay - vy$ is $= bu - ux$. Hence, by residual division, we have $a[x \,\dot{-}\, y] - v\,[x \,\dot{-}\, y] - y\,[x \,\dot{-}\, v] = b\,[x \,\dot{-}\, u] - x\,[x \,\dot{-}\, u] - u$; and consequently, by substitution, $a - v - \frac{uy}{b} \times [x \,\dot{-}\, y] = \overline{b - x} \times \frac{a - v}{b} \times [x \,\dot{-}\, y] - u$. From whence it appears, that $\frac{uy}{b} \times [x \,\dot{-}\, y]$ is $= \frac{x \times \overline{a - v}}{b} \times [x \,\dot{-}\, y] - u$; therefore $[x \,\dot{-}\, y]$ is $= \frac{bu}{uy - x \times \overline{a - v}}$. Consequently $\frac{y}{[x \,\dot{-}\, y]}$, the value of the subtangent BN, is $= \frac{y^2}{b} - \frac{xy \times \overline{a - v}}{bu} = \frac{y^2 + x^2 - bx}{b}$.

7. AqPq

7.

AqPq being a curve of the ſpiral kind, whoſe ordinates Cq, CP, Cq, all iſſue from the point C; let the circular arc $defe$ be deſcribed, whoſe radius is 1; and draw any right line Cd, interſecting the ſaid arc in d. Then, ſuppoſing the right line NrPr to touch the curve in P; and ſuppoſing Cqr to interſect the ſaid tangent in r, and the circular arc in e: if CP be called y; Cq, $y_{,}$; the ſine of the angle dCf, (to the radius 1,) u; the ſine of dCe, $u_{,}$; and the ſine and coſine of CPN, s and c reſpectively: the ſine of eCf will be denoted by $u\sqrt{1-u_{,}^2} - u_{,}\sqrt{1-u^2}$ or $u_{,}\sqrt{1-u^2} - u\sqrt{1-u_{,}^2}$, according as de is leſs or greater than df; and the ſine of CrN, (or CrP,) by $cu\sqrt{1-u_{,}^2} - cu_{,}\sqrt{1-u^2} - suu_{,} + s\sqrt{1-u^2} \times \sqrt{1-u_{,}^2}$, which laſt ſine, for brevity ſake, we will call k: Moreover Cr will be $= \frac{sy}{k}$, and the reſidual Cr — Cq $(= qr) = \frac{sy}{k} - y_{,} = \frac{sy - ky_{,}}{k}$. Now, C$qr$ being drawn on either ſide of CP, Cr is manifeſtly greater than Cq: therefore $\frac{sy-ky_{,}}{k}$ (the value of Cr — Cq) muſt be always poſitive, when, u being of any given value, $u_{,}$ is either leſs or greater than u. Fig. 12.

If the convexity of the curve be towards C, as in Fig. 13. $\frac{sy-ky_{,}}{k}$ (the value of Cr — Cq) will be always negative, when, u being of any given value, $u_{,}$ is either leſs or greater than u.

Since the expreſſion $\frac{sy-ky_{,}}{k}$ muſt be always poſitive or always negative, when, u being of any given value, $u_{,}$ is either leſs or greater than u; the quotient of that expreſſion divided by $u - u_{,}$ (viz. $\frac{sy-ky_{,}}{k.\overline{u-u_{,}}}$) will, it is obvious, be poſitive when $u_{,}$ is leſs than

 u,

u, and negative when $u_{,}$ is greater than u; or poſitive when $u_{,}$ is greater than u, and negative when $u_{,}$ is leſs than u, how near ſo-ever $u_{,}$ be taken to u. Therefore, by Chap. 4. Art. 1. the value of that quotient, when $u_{,}$ is equal to u, (i. e. when $y_{,}$ is $= y$,) or the reciprocal of that value, will be $= 0$. But, by the method taught in Chap. 2. Corollary to Art. 6. the value of $\frac{sy - ky_{,}}{k \cdot u - u_{,}}$, when $u_{,}$ is $= u$, is found equal to $\overline{s\,[u \perp y] - \frac{cy}{\sqrt{1-u^2}}} \div s$.

Therefore $s\,[u \perp y] - \frac{cy}{\sqrt{1-u^2}}$ is $= 0$*: Hence $\frac{s}{c} = \frac{1}{\sqrt{1-u^2}} \times \frac{y}{[u \perp y]}$.

If the arc df be called w, $\frac{1}{\sqrt{1-u^2}}$ will be $= [u \perp w]$, by Art. 5. It therefore appears, by ſubſtitution, that

$$\frac{y\,[u \perp w]}{[u \perp y]} \text{ is } = \frac{s}{c} = \text{Tang. of the angle CPN.}$$

Suppoſe CN, perpendicular to the ray CP, to interſect the tangent PN in N: then, radius being to CP, as the tangent of CPN to CN, CN will be $= \frac{y^2\,[u \perp w]}{[u \perp y]}$.

Seeing c is $= \sqrt{1 - s^2}$, it is evident that $\frac{s}{\sqrt{1-s^2}}$ is $= \frac{y\,[u \perp w]}{[u \perp y]}$: from whence we have $s = \frac{y\,[u \perp w]}{\sqrt{[u \perp y]^2 + y^2[u \perp w]^2}}$.

8.

Fig. 12. When the curve AqP is concave towards C, and the angle CPN, made by the ray CP and tangent PN, is acute; the ray Cqr being drawn, interſecting the ſaid tangent in r, Pr will be

* The other equation (viz. $\frac{s}{s\,[u \perp y] - \frac{cy}{\sqrt{1-u^2}}} = 0$) which, if poſſible, would follow from what is ſaid above, is manifeſtly impoſſible.

leſs

lefs or greater than the curve Pq, according as de is lefs or greater than df: And when the curve AqP is convex towards C, and the angle CPN is acute, as in Fig. 13. Pr will be greater or lefs than the curve Pq, according as de is lefs or greater than df. This may be eafily proved, by reafoning as in Art. 4.

Now, retaining the notation in the laft Art. Pr will be equal to $\frac{y}{k}\times\overline{u\sqrt{1-u_{,}^2}-u_{,}\sqrt{1-u^2}}$ or $-\frac{y}{k}\times\overline{u\sqrt{1-u_{,}^2}-u_{,}\sqrt{1-u^2}}$, according as de is lefs or greater than df. Therefore, calling the parts AP, Aq, of the curve AqPq, z and $z_{,}$ refpectively; it follows that the expreffion $z-z_{,}-\frac{y}{k}\times\overline{u\sqrt{1-u_{,}^2}-u_{,}\sqrt{1-u^2}}$ muft be always pofitive or always negative, when, u being of any value whatever, $u_{,}$ is either greater or lefs than u. It is obvious then, that the quotient of the faid expreffion, divided by $u-u_{,}$ (viz. $\frac{z-z_{,}}{u-u_{,}}-\frac{y}{k}\times\frac{u\sqrt{1-u_{,}^2}-u_{,}\sqrt{1-u^2}}{u-u_{,}}$) will be pofitive when $u_{,}$ is lefs than u, and negative when $u_{,}$ is greater than u; or pofitive when $u_{,}$ is greater than u, and negative when $u_{,}$ is lefs than u, how near foever $u_{,}$ be taken to u. Therefore, by Chap. 4. Art. 1. the value of that quotient, when $u_{,}$ is equal to u, (i.e. when $y_{,}$ is $=y$, and $z_{,}=z$,) or the reciprocal of that value, will be $=0$. But, by the method taught in Chap. 2. Cor. to Art. 6. the value of $\frac{u\sqrt{1-u_{,}^2}-u_{,}\sqrt{1-u^2}}{u-u_{,}}$, when $u_{,}$ is $=u$, is found equal to $\frac{u^2}{\sqrt{1-u^2}}+\sqrt{1-u^2}=\frac{1}{\sqrt{1-u^2}}$; and k is then equal to s. Therefore $[u \perp z]-\frac{y}{s\sqrt{1-u^2}}$ is $=0$*: Hence $[u \perp z]=\frac{y}{s\sqrt{1-u^2}}$.

* The other equation (viz. $\frac{1}{[u \perp z]-\frac{y}{s\sqrt{1-u^2}}}=0$) which, if poffible, would follow from what is faid above, is impoffible.

Seeing

Seeing that $\frac{1}{\sqrt{1 - u^2}}$ is $= [u \dot{-} w]$, by Art. 5. it appears, by ſubſtitution, that $[u \dot{-} z]$ is $= \frac{y[u \dot{-} w]}{s} = \sqrt{[u \dot{-} y]^2 + y^2 [u \dot{-} w]^2}$; s being $= \frac{y[u \dot{-} w]}{\sqrt{[u \dot{-} y]^2 + y^2[u \dot{-} w]^2}}$, by the preceding Article.

From what is ſaid it is evident, that s is $= \frac{y[u \dot{-} w]}{[u \dot{-} z]}$; and conſequently c $(= \sqrt{1 - s^2})$ is $= \frac{[u \dot{-} y]}{[u \dot{-} z]}$.

Moreover, ſuppoſing CQ to be perpendicular to the tangent PQ; radius will be to y, as $\frac{y[u \dot{-} w]}{[u \dot{-} z]}$ $(= s)$ to CQ; therefore CQ will be $= \frac{y^2[u \dot{-} w]}{[u \dot{-} z]}$: and radius will be to y, as $\frac{[u \dot{-} y]}{[u \dot{-} z]}$ $(= c)$ to PQ; therefore PQ will be $= \frac{y[u \dot{-} y]}{[u \dot{-} z]}$.

Corollary. Hence it appears, that the perpendicular CQ, the tangent PQ, and the ray CP, are to each other as $y\,[u \dot{-} w]$, $[u \dot{-} y]$, and $[u \dot{-} z]$ reſpectively.

9.

By means of the theorems inveſtigated above, we are now enabled to draw tangents to any ſpiral whoſe equation is given.

Fig. 14. Example I. *To draw a tangent to the ſpiral of* Archimedes; *whoſe nature is ſuch, that, any circle* AF *being deſcribed about the center* C, *and any ray* CfFP *being drawn, the arc* AF *is to* FP *in a conſtant ratio.*

Let CA be $= m$, C$d = 1$, the arc $df = w$, CP $= y$; and let the given ratio of AF to FP be as m to n: Then mw will be $=$ AF, and $nw = y - m$. From whence we have $n\,[u \dot{-} w] = [u \dot{-} y]$; and conſequently, by ſubſtitution, CN $(= \frac{y^2[u \dot{-} w]}{[u \dot{-} y]}$, by Art. 7.) is found $= \frac{y^2}{n}$.

Example

EXAMPLE II. *To draw a tangent to the ſpiral* CdP; *whoſe nature is ſuch, that the arc* CdP *is to the ray* CP *in an invariable ratio.* Fig. 15.

Let CdP be $= z$, CP $= y$; and let the invariable ratio of z to y be that of a to b: Then bz will be $= ay$. From whence we get $\frac{a}{b}[u \dot{-} y] = [u \dot{-} z]$, which, by the laſt Article, is $= \sqrt{[u \dot{-} y]^2 + y^2 [u \dot{-} w]^2}$. Hence it appears, that $[u \dot{-} y]$ is $= \frac{b}{\sqrt{a^2 - b^2}} \times y[u \dot{-} w]$; and therefore $\frac{y[u \dot{-} w]}{[u \dot{-} y]}$, the tangent of the angle CPN, is $= \frac{\sqrt{a^2 - b^2}}{b}$; and CN $= \frac{\sqrt{a^2 - b^2}}{b} \times y$.

It is obſervable, that the angle made by the tangent PN with the ray CP is invariable; which is a known property of the *logarithmic ſpiral.*

10.

Suppoſe the moveable curve aB *to revolve along the immoveable curve* AB, *ſo that the arcs* aB, AB *be always equal*; *and ſuppoſe, that, during the motion, the point* P, *having a certain poſition with reſpect to the curve* aB, *deſcribes the curve* OPQ, *the curves and the deſcribing point keeping always in the ſame plane: then, if from* B, *the point where the two curves*, aB, AB, *touch each other when the deſcribing point is in* P, BP *be drawn*; PR, *perpendicular thereto, ſhall touch the curve* OPQ *in* P. Fig. 16.

For, about the center B, with the radius BP, deſcribe the circular arc EPF: and, having drawn BFQ, ſuppoſe that, when the deſcribing point is in Q, the moveable curve is poſited in *a*bD; b being that point thereof which was at B when the deſcribing point was at P, and D being now the point of contact of the two curves *a*bD, ABD: join bB, bQ; and let *ef* touch the curve *a*bD in b, and meet the curve ABD in *f*.—Then, becauſe the arcs BD, bD are equal; and b*f* + *the arc* D*f* is greater than the arc bD, b*f* ſhall be greater than the arc B*f*, and conſequently ſtill greater than the chord B*f*: Wherefore the angle bB*f* (made by bB and the chord B*f*) will be greater than Bb*f*, and Bb*e* greater than bBg, made by bB and the conti- Fig. 17.

continuation of the chord fB. It is evident therefore, that the angle BbQ, which is = Bbe + ebQ, will be greater than QBb, which is = bBg — QBg; and consequently BQ will be greater than bQ. But bQ is manifestly equal to BP, being the same line transferred with the moveable curve. Therefore BQ is greater than BP, i. e. than BF. Hence it appears, that the point Q is without the circular arc EPF.

Fig. 18. Suppose now, that, when the describing point is on the other side of P in O, the moveable curve is posited in adb; b being that point thereof which coincides with B when the describing point is in P, and d being now the point of contact of the two curves adb, AdB: and join BO, Bb, bO, bd.—Then, the arc Bd, which is equal to the arc bd, will be greater than the chord bd. But, Be being drawn, touching the curve AdB in B, and intersecting the chord bd in e, Be + ed will be greater than the arc Bd; and therefore Be will be greater than be: Wherefore the angle Bbe will be greater than bBe. It is plain then, that the angle BbO, which is = Bbe + ebO, will be greater than bBO, which is = bBe — eBO; and consequently BO will be greater than bO. But bO is evidently equal to BP, being the same line in a different position. Therefore BO is greater than BP, i. e. than BE. Hence it appears, that the point O is without the circular arc EPF: Therefore, the same being proved of the point Q, it follows, that the said circular arc touches the curve OPQ in P. Consequently PR, which is a tangent to that circular arc, will also touch the curve OPQ in P.

Whatever position the describing point P may have with respect to the moveable curve, and whether that curve revolves along the convexity or concavity of the immoveable one, the tangents to the curve described by P will always be determined by the rule here given; and in no case will the demonstration differ greatly from the above.—The describing point may indeed be so posited with respect to the moveable curve, that the circular arc EPF shall sometimes fall without the curve OPQ, which occasions some little difference in the demonstration; but there will not then be any particular difficulty in it: Any farther explanation is therefore unnecessary.

This article (which was suggested by one in Huygen's Horolog. Oscillator.) *will be found of great use in many enquiries concerning* cycloidal curves.

Scho-

SCHOLIUM. If the point P, inſtead of keeping a certain po- Fig. 19.
ſition with reſpect to the curve aB, be ſuppoſed to move in ſuch a manner along the baſe MN of that curve, whilſt the curve itſelf revolves as before-mentioned, that any ordinate being drawn from the point of contact B perpendicular to the ſaid baſe, P ſhall always be at the end of the correſpondent abſciſſa; then, by what is proved above, will that abſciſſa (or the baſe) be a tangent to the curve OPQ deſcribed by the point P during ſuch motion.

11.

With reſpect to curves referred to an axis, it is obvious, that Fig. 1.
when the convexity of the curve is downwards, as in Fig. 1. the 2.
value of the quotient of the ordinate divided by the ſubtangent increaſes as the abſciſſa (x) is taken greater and greater: and when the convexity is upwards, as in Fig. 2. the value of the ſaid quotient decreaſes as x is taken greater and greater. Now, by what is ſaid above, that quotient (reſuming the notation in Art. 1.) is equal to $[x \perp y]$. Therefore, in the former caſe, $[x \perp y] - [x_{,} \perp y_{,}]$ will be poſitive when $x_{,}$ is leſs than x, and negative when $x_{,}$ is greater than x: and, in the latter caſe, $[x \perp y] - [x_{,} \perp y_{,}]$ will be poſitive when $x_{,}$ is greater than x, and negative when $x_{,}$ is leſs than x. It is evident then, that, $x_{,}$ being either leſs or greater than x, $\dfrac{[x \perp y] - [x_{,} \perp y_{,}]}{x - x_{,}}$ will be always poſitive or always negative, according as the convexity is downwards or upwards. Hence it is plain, that the value of $[x \perp y]$ (ſuppoſing both it and its reciprocal to be finite) will be poſitive or negative, according as the convexity of the curve is downwards or upwards.

12.

With regard to curves referred to a fixed point C, it is evident, Fig. 12.
that, when the concavity of the curve is towards C, as in Fig. 12. 13.
the perpendicular from C on the tangent increaſes as the ray or ordinate (y) is taken greater and greater: and when the convexity is towards C, as in Fig. 13. the ſaid perpendicular decreaſes

as

as y is taken greater and greater. But, by what is said above, that perpendicular (resuming the notation in Art. 7. and 8.) is equal to $\frac{y^2[u \dot{-} w]}{[u \dot{-} z]}$. Consequently, in the former case, $\frac{y^2[u \dot{-} w]}{[u \dot{-} z]} - \frac{y_{,}^2[u_{,} \dot{-} w_{,}]}{[u_{,} \dot{-} z_{,}]}$ will be positive when $u_{,}$ is less than u, and negative when $u_{,}$ is greater than u: and, in the latter case, $\frac{y^2[u \dot{-} w]}{[u \dot{-} z]} - \frac{y_{,}^2[u_{,} \dot{-} w_{,}]}{[u_{,} \dot{-} z_{,}]}$ will be positive when $u_{,}$ is greater than u, and negative when $u_{,}$ is less than u. Therefore it is manifest, that, $u_{,}$ being either less or greater than u, $\overline{\frac{y^2[u \dot{-} w]}{[u \dot{-} z]} - \frac{y_{,}^2[u_{,} \dot{-} w_{,}]}{[u_{,} \dot{-} z_{,}]}} \div \overline{u - u_{,}}$ will be always positive or always negative, according as the concavity or convexity is towards C. From whence it appears, that the value of $\left[u \dot{-} \frac{y^2[u \dot{-} w]}{[u \dot{-} z]}\right]$ (supposing both it and its reciprocal to be finite) will be positive or negative, according as the curve is concave or convex towards C.

THE

THE

RESIDUAL ANALYSIS.

CHAP. VI.

Of the Investigation of useful Theorems, *by finding the nature of a Curve from a given property of its* Tangents.

I.

BF *Being parallel to* AE; *it is proposed to find a curve line, such, that,* ab *being a tangent thereto, at any point thereof,* $\overline{\mathrm{Bb}}|^2 - \overline{\mathrm{Aa}}|^2$ $(= \overline{\mathrm{Bb} + \mathrm{Aa}} \times \overline{\mathrm{Bb} - \mathrm{Aa}})$ *shall always be equal to the invariable square* c^2: *which tangent is supposed to intersect* AE *and* BF *in the points* a *and* b *respectively.* Fig. 20.

Supposing p to be the point where ab touches the required curve, let mp be drawn parallel to AE; and call AB, a; Am, x; mp, y.

Then will Aa be $= y - x\,[x \dot{-} y]$,

Bb $= y + \overline{a - x} \times [x \dot{-} y]$,

and $\overline{\mathrm{Bb}}|^2 - \overline{\mathrm{Aa}}|^2 = \overline{a^2 - 2ax} \times [x \dot{-} y]^2 + 2ay\,[x \dot{-} y] = c^2$.

Hence,

Hence, by residual division, we get

$$\overline{a^2 - 2ax} \times [x \dot{-} y]^2 \times [x \dot{-} y] + ay\,[x \dot{-} y] = 0\,;$$

from whence we have $[x \dot{-} y] = \frac{y}{2x - a}$.

Which last quantity being substituted for its equal in the value of c^2, we find

$$y^2 = \frac{c^2}{a} \times \overline{2x - a}\,.$$

The curve therefore is a Parabola; AB coincides with a diameter thereof, the parameter of which is $= \frac{2c^2}{a}$; and, V being the point where that diameter meets the curve, AV is $=$ BV.

Corollary. AB being a diameter of any conical Parabola, and aAa′ parallel to the corresponding ordinates; if Aa on one side of A be equal to Aa′ taken on the contrary side of A, the tangents ap, a′p′ will always intersect each other in the right line Bb, being that ordinate continued whose abscissa VB is $=$ VA.

2.

Fig. 21. BF *being parallel to* AE, *it is proposed to find a curve line, such, that,* ab *being a tangent thereto, at any point thereof,* $\overline{\mathrm{Bb}}]^2 + \overline{\mathrm{Aa}}]^2$ *shall always be equal to the invariable square* c^2.

Let the lines AB, A*m*, *mp* be as in the preceding article: Then Aa being $= y - x\,[x \dot{-} y]$, and Bb $= y + \overline{a - x} \times [x \dot{-} y]$; $\overline{\mathrm{Bb}}]^2 + \overline{\mathrm{Aa}}]^2$ is $= 2 \times \overline{y - x\,[x \dot{-} y]}\,]^2 + \overline{a^2 - 2ax} \times [x \dot{-} y]^2 + 2ay\,[x \dot{-} y] = c^2$.

Hence, by residual division, and dividing by $2\,[x \dot{-} y]$, we get

$$2x^2\,[x \dot{-} y] - 2xy + \overline{a^2 - 2ax} \times [x \dot{-} y] + ay = 0\,;$$

from whence we have $[x \dot{-} y] = \frac{ay - 2xy}{2ax - a^2 - 2x^2}$.

Which

Which laſt quantity being ſubſtituted for its equal in the value of c^2, it appears that the equation of the curve is

$$y^2 = \frac{c^2}{a^2} \times \overline{a^2 - 2ax + 2x^2}:$$

Anſwering to an Hyperbola; whereof CV equal to $\frac{c}{\sqrt{2}}$, and parallel to AE and BF, is a ſemi-diameter; whoſe ſemi-conjugate is $AC = BC = \frac{a}{2}$; C being the center.

As by inveſtigating the firſt propoſition, we diſcovered a remarkable property of the Parabola; ſo here we diſcover a property of the Hyperbola, equally remarkable: And it is obvious, that a Corollary may be drawn from what is done in this article, ſimilar to that in the preceding.

3.

BF *being ſtill parallel to* AE; *it is propoſed to find a curve line, ſuch, that,* ab *being a tangent thereto, at any point thereof, the rectangle* Aa × Bb *ſhall always be equal to the invariable ſquare* c^2. Fig. 22. 23.

The lines AB, A*m*, *mp* being as in the preceding articles; Aa will be $= y - x[x \perp y]$, and $Bb = a[x \perp y] \pm \overline{y - x[x \perp y]}$.

Therefore Aa × Bb will be $= a[x \perp y] \times \overline{y - x[x \perp y]} \pm \overline{y - x[x \perp y]}\vert^2 = c^2$.

Hence, by means of our reſidual diviſion, we get

$$\overline{a \mp 2x} \times \overline{y - x[x \perp y]} - ax[x \perp y] = 0;$$

from whence we have $[x \perp y] = \frac{ay \mp 2xy}{2ax \mp 2x^2}$.

Conſequently, by ſubſtitution, we find

$$Aa = \frac{ay}{2a \mp 2x}, \quad Bb = \frac{ay}{2x}, \quad \text{and} \quad \frac{a^2y^2}{4 \cdot \overline{ax \mp x^2}} = c^2.$$

Therefore, in the firſt caſe, the equation being $y^2 = \frac{4c^2}{a^2} \times \overline{ax - x^2}$, the curve is a Circle or an Ellipſis; and in the ſecond caſe, the equation being $y^2 = \frac{4c^2}{a^2} \times \overline{ax + x^2}$, the curve is an Hyperbola. AE, BF touch the conic ſection, or oppoſite hyperbolas, in A and

and B: AB being a diameter of the section; whose conjugate is equal to $2c$, and parallel to those tangents.

Corollary I. It appears that c, the semi-conjugate to the diameter AB, is a mean proportional between Aa and Bb.

Corollary II. x being to $\frac{y}{2}$ as a to $\frac{ay}{2x}$, the value of Bb; it follows, that if AE, BF be parallel tangents to any ellipsis, or opposite hyperbolas, and mp any ordinate to the diameter AB, a line drawn from the point of contact A, so as to bisect mp, will always meet the tangent from p in the line BF.

Corollary III. Let $a'b'$ be another tangent to the curve. Then, Aa' being to Bb as Aa to Bb', the right lines $b'a$, ba' will meet in the diameter AB, or in the continuation thereof.

Scholium. If it be required to describe, to a given center, a conic section, or opposite hyperbolas, that shall touch three right lines given by position: Draw a line parallel to one of the given tangents, so that the given center shall be between them and equidistant from each, which line it is plain must be another tangent to the required curve; and then the diameter corresponding to the two points of contact of the parallel tangents may be readily found by this Corollary, and its conjugate will be known by Cor. I. The description of the curve then easily follows.

Corollary IV. By considering BF to be removed to an infinite distance from AE, we may infer from the last Corollary, that, if AE, ab, $a'b'$ be tangents to any conical parabola, right lines drawn through a and a' parallel to $a'b'$, ab respectively, will meet in the continuation of that diameter AB which passes through the point where AE touches the curve.

Scholium. This Corollary, it is obvious, enables us to describe, with great facility, a parabola that shall touch three right lines given by position, and have its axis parallel to another right line given by position; and likewise to describe a parabola, that shall touch four right lines given by position.

4. AB,

4.

AB, BC *making any angle at* B; *it is propoſed to find a curve line, ſuch, that,* ab *being a tangent thereto, at any point thereof, the rectangle* Aa × bC *ſhall always be equal to the invariable ſquare* c^2. Fig. 24. 25.

Suppoſing p to be the point where ab touches the required curve, let AD, mp be drawn parallel to BC, AB reſpectively; and call AB, a; BC, b; Am, x; mp, y.

Then will Aa be $= y - x[x \dot{-} y]$, Ba $= a - \overline{y - x[x \dot{-} y]}$,

$$\text{Bb} = \frac{a - \overline{y - x[x \dot{-} y]}}{[x \dot{-} y]}, \text{ and bC} = b \backsim \frac{a - \overline{y - x[x \dot{-} y]}}{[x \dot{-} y]}.$$

Therefore

$$\text{Aa}\times\text{bC is} = b \times \overline{y - x[x \dot{-} y]} \backsim \frac{a \times \overline{y - x[x \dot{-} y]} - \overline{y - x[x \dot{-} y]}^2}{[x \dot{-} y]} = c^2.$$

Hence, after multiplying by $[x \dot{-} y]$, we, by means of our reſidual diviſion, readily find

$$[x \dot{-} y] = \frac{ax + by - 2xy \mp c^2}{2bx - 2x^2}.$$

Conſequently, by a proper ſubſtitution, it appears that the equation of the curve is

$$\overline{ax + by \mp c^2}^2 = \overline{4ab \mp 4c^2} \times xy:$$

Which correſponds to an Ellipſis, or Hyperbola. And if CD be parallel to BA; AB, BC, CD, and DA will be tangents to the ellipſis, or oppoſite hyperbolas. Moreover, taking AE and CF each equal to $\frac{c^2}{b}$, the right line EF will be a diameter of the Figure, whoſe conjugate diameter will be $= \frac{2c}{b} \times \sqrt{ab \mp c^2}$.

COROLLARY. The middle point P of the right line AC is the center of the Figure: And, (ſuppoſing $\acute{a}\acute{b}$ another tangent to the curve,) Aa being to C$\acute{b}$, as A$\acute{a}$ to Cb; if a$\acute{b}$, $\acute{a}$b be joined, and thoſe lines biſected, a right line drawn through the points of biſection will paſs through the point P.

For

Fig. 26. For let Bd be to BC, as Aa to Cb′; and parallel to Cd draw b′e,
27. bf; alſo, having biſected thoſe three parallels in *m*, *n*, and *o*, and the three lines AC, ab′, a′b in *p*, *q*, *r* reſpectively, draw *mp*, *nq*, *or*, which will all three be parallel to AB. Then from the analogies

$$\text{de} : \text{Cb}' :: \text{Bd} : \text{BC} :: \text{Aa} : \text{Cb}',$$

$$\text{and df} : \text{Cb} :: \text{Bd} : \text{BC} :: \text{Aa}' : \text{Cb},$$

it appears that de will be = Aa, and df = Aa′.

Therefore, in caſe the firſt (Fig. 26.) ae and a′f will each be equal to Ad; and the parallels *mp*, *nq*, *or* each equal to $\frac{1}{2}$Ad. Conſequently, *mno* being a right line, *pqr* is, in this caſe, a right line parallel thereto.

Moreover, in caſe the ſecond (Fig. 27.) ae and a′f will be equal to Ad + 2de and Ad + 2df reſpectively; and the parallels *mp*, *nq*, *or* equal to $\frac{1}{2}$Ad, $\frac{1}{2}$Ad + de, and $\frac{1}{2}$Ad + df reſpectively. It is evident therefore, that, in this caſe, *pqr* is a right line parallel to dC.

Scholium. If it be required to deſcribe an ellipſis, or oppoſite hyperbolas, that ſhall touch five right lines given by poſition; the center of the Figure may be eaſily found by this Corollary: And then we may proceed according to the Scholium to Cor. 3. of the laſt article.

5.

Fig. 28. AB, BC *making any angle at* B; *it is propoſed to find ſuch a curve line, that,* ab *being a tangent thereto, at any point thereof,* Aa × bC *ſhall always be to* Ba × Bb *in the invariable ratio of* a^2 *to* c^2.

Lines being drawn and denominated as in the preceding article; we have, from what is there ſaid, and from the given ratio of the rectangle Aa × bC to the rectangle Ba × Bb,

$$c^2 \times \overline{\overline{b[x \dot{-} y] - a} \times \overline{y - x[x \dot{-} y]} + \overline{y - x[x \dot{-} y]}|^2} =$$
$$a^2 \times \overline{a - y + x[x \dot{-} y]}|^2.$$

Hence,

Hence, by means of our residual division, ($x - \dot{x}$ being the divisor,) we get

$$[x \dot{-} y] = \frac{bc^2y - a\,.\,\overline{2a^2 - c^2}\,.\,x + \overline{2a^2 - 2c^2}\,.\,xy}{2bc^2x + \overline{2a^2 - 2c^2}\,.\,x^2}.$$

Which last expression being substituted for its equal in the equation above, the equation of the curve will be obtained. Or the same may be found (perhaps more readily) by proceeding as follows.

Let $\frac{1}{[y \dot{-} x]}$ be substituted for its equal $[x \dot{-} y]$, in the first equation, and there will result

$$c^2 \times \overline{b - a\,[y \dot{-} x] \times \overline{y\,[y \dot{-} x] - x} + \overline{y\,[y \dot{-} x] - x}}\,^2 = a^2 \times \overline{\overline{a - y}\,.\,[y \dot{-} x] + x}\,^2.$$

From whence, by residual division, ($y - \dot{y}$ being the divisor,) we find

$$[y \dot{-} x] = \frac{bc^2y - a\,.\,\overline{2a^2 - c^2}\,.\,x + \overline{2a^2 - 2c^2}\,.\,xy}{2a^4 - 2a\,.\,\overline{2a^2 - c^2}\,.\,y + \overline{2a^2 - 2c^2}\,.\,y^2}.$$

Therefore, $[x \dot{-} y]$ being $= \frac{1}{[y \dot{-} x]}$,

$$\frac{bc^2y - a\,.\,\overline{2a^2 - c^2}\,.\,x + \overline{2a^2 - 2c^2}\,.\,xy}{2bc^2x + \overline{2a^2 - 2c^2}\,.\,x^2} \text{ will be} = \frac{2a^4 - 2a\,.\,\overline{2a^2 - c^2}\,.\,y + \overline{2a^2 - 2c^2}\,.\,y^2}{bc^2y - a\,.\,\overline{2a^2 - c^2}\,.\,x + \overline{2a^2 - 2c^2}\,.\,xy}.$$

And the equation of the curve, from thence found, is

$$a^2c^2x^2 + b^2c^2y^2 + 2ab\,.\,\overline{2a^2 - c^2}\,.\,xy - 4a^4bx = 0;$$

which corresponds to an Ellipsis or Hyperbola, according as c is greater or less than a.—It appears that AB, and BC will touch the Figure in A and C: And that if CE be parallel to BA, and equal to $\frac{2a^2 - c^2}{c^2}\,.\,a$; AF, parallel to BE, will coincide with a diameter of the conic section; which diameter will be equal to $\frac{c^2}{c^2 \backsim a^2}$BE, and its conjugate equal to $\frac{2a^2}{\sqrt{c^2 \backsim a^2}}$.

COROLLARY I. This conclusion suggests some remarkable properties of the conic sections; and also an easy method of describing

ſcribing elliptical or hyperbolic trajectories, that ſhall touch right lines given by poſition, in certain caſes, to which the theorems in the preceding articles are not ſo readily, if at all applicable.

EXAMPLE. *To deſcribe a trajectory that ſhall touch three right lines given by poſition, and two of them in given points.*

Fig. 29. Let AB, BC, ab be the three given lines, and A and C the points of contact of the two firſt of thoſe lines.

On the line BA take Ae equal to AB, and parallel thereto draw Cf; alſo, having drawn eb$f$, draw fBg meeting Cag in g: then will gA continued coincide with a diameter of the required trajectory. In the ſame manner may the direction of the diameter from C be found; and conſequently the center of the conic ſection, it being the point where thoſe diameters interſect each other.—The buſineſs then may be eaſily completed, after the manner commonly taught by the writers on Conics.

This conſtruction is ſo eaſily inferred from what is done above, that I think it unneceſſary to be more explicit.

Fig. 28. COROLLARY II. If c be equal to a, the required curve will be a Parabola. It therefore evidently follows, that, any two tangents AB, BC being drawn to any conical parabola ApC, touching the ſame at A and C, and interſecting each other at B; if any third tangent ab be drawn (to the ſame parabola) interſecting the tangents AB, BC, at a and b reſpectively, Aa × Cb will be = Ba × Bb. And a knowledge of this remarkable property of the parabola enables us to find ſome others, and readily to perform the buſineſs mentioned in the Scholium to Cor. 4. Art. 3.

The ſolutions to theſe problems being eaſily obtained by means of our reſidual diviſion, and the firſt theorem in the preceding chapter, without any farther knowledge of our doctrine; they are, it is preſumed, not improperly inſerted here.—In a ſubſequent chapter, we ſhall ſhew how, by a different artifice in our Calculus, ſome other theorems relating to curve lines may be inveſtigated from given properties of their tangents.

Some of the theorems here inveſtigated may be ſeen, demonſtrated in a different manner, in *Sir* ISAAC NEWTON'*s Philoſ. Natur. Princ. Mathem.*

THE

THE

RESIDUAL ANALYSIS.

CHAP. VII.

Of the Evolution *and* Curvature *of Lines*; *with ſome Inferences relating to the* Focuſes *of reflected and refracted rays, and the curves call'd* Cauſtics.

I.

A Perfectly flexible thread, $dCaA$, being applied along the convexity of the curve dCa, from d to a; ſuppoſe the part aA (of ſuch thread) to be extended in a right line that touches the ſaid curve in a; ſuppoſe alſo, that, whilſt one end of the thread remains fixed at d, the other end A be moved towards D, (in the ſame plane with the curve,) ſo that the thread be continually unwrapped from the curve, and the part CP which is diſengaged therefrom be always extended in a right line that touches the curve: then ſhall the point A trace the *involute* curve APD that is ſaid to be deſcribed by the evolution of aCd, which is itſelf called the *evolute*, and the right line CP is called the *radius of evolution* correſponding to the point P. Fig. 30.

2.

Let Pr be drawn at right angles to CP; and, with the radius CP, deſcribe the circle EPF. Draw any other radius of evolu-

 tion

tion eq, e being between a and C; join Ce; and draw Cqr, interſecting Pr in r. Then, by the nature of the evolution, eq + *the curve* Ce being manifeſtly = CP, eq + *the chord* Ce is leſs than CP; conſequently the right line Cq is ſtill leſs than CP. But Cr is greater than CP: therefore Cq is leſs than Cr, and the involute AP is wholly between the right line Pr and the evolute aC.

The point e being on the other ſide of C, and g being the point where PC continued interſects er; eq (= CP + *the curve* Ce) is leſs than eg + gP: therefore gq is leſs than gP. But gr is greater than gP: therefore gq is leſs than gr, and the involute Pq is wholly between the right line Pr and the evolute Ce. Conſequently both Pr, and the circle EPF, touch the involute in P.

The point e being between C and d, ſuppoſe F to be the point where the circle EPF interſects eqr. Then eq (= CP + *the curve* Ce = CF + *the curve* Ce) will be greater than eF: conſequently the circular arc PF falls within the involute Pq; and it is plain, that no circle deſcribed through P, with a radius leſs than CP, can paſs between PF and Pq. Moreover, gq being leſs than gP, (as before obſerved,) a circle deſcribed from any point g through P, with a radius gP greater than CP, paſſes without Pq and PF. Therefore no circle deſcribed through P can paſs between Pq and PF.

The point e being between C and a, CP (as is proved above) is greater than Cq: conſequently the circular arc PE falls without the involute Pq; and it is evident, that no circle deſcribed through P, with a radius greater than CP, can paſs between PE and Pq. Moreover, f being the point where qe continued interſects CP, Pf + fC (= qe + *the curve* eC) being leſs than qf + fC, Pf is leſs than qf: conſequently a circle deſcribed from any point f through P, with a radius fP leſs than CP, paſſes within Pq and PE. It follows therefore, that no circle deſcribed through P can paſs between the circle EPF and the involute curve APD, all other circles paſſing either within or without both the ſaid circle and curve.

3.

The circle which touches a curve ſo cloſely, that no circle can be drawn through the point of contact between them, is ſaid to have the ſame curvature with the curve at that point. Which circle

circle is called the *circle of curvature*; its center, the *center of curvature*; and its semidiameter, the *radius of curvature*, corresponding to such point of contact.

It appears then, that the circle EPF (whose center is C) is the circle of curvature of the involute APD, at P. Therefore, by considering any curve as an involute, the radius of curvature (or evolution), at any point thereof, may be readily found, as in the following articles.

4.

The curve APD being referred to a base AK, suppose aCd to be the curve by whose evolution APD is described: draw CH parallel to KA; and suppose GH, parallel to the ordinate PB, to intersect the said base in I. Call AB, x; BP, y; the curve AP, z; the tangent NP, t; the subtangent BN, s; AI, b; GI, c; GH, v; CH, u; the curve aC, w; and (CP) the radius of curvature at P, R. Fig. 31.

Then, by similar triangles and Chap. 5. Art. 5.

$s : t :: v + y + c : \mathrm{R} :: 1 : [v \overset{\prime}{-} \mathrm{R}]$ $(= [v \overset{\prime}{-} w])$.

Therefore 1 is $= \frac{s}{t} \times [v \overset{\prime}{-} \mathrm{R}]$, and $\frac{\mathrm{R}s}{t} - y - c = v$.

From the last equation, we get, by residual division,

$$\frac{s[v \overset{\prime}{-} \mathrm{R}]}{t} + \frac{\mathrm{R}[v \overset{\prime}{-} s]}{t} - \frac{\mathrm{R}s\,[v \overset{\prime}{-} t]}{t^2} - [v \overset{\prime}{-} y] = 1 = \frac{s[v \overset{\prime}{-} \mathrm{R}]}{t}.$$

Hence R is found $= \frac{t^2[v \overset{\prime}{-} y]}{t[v \overset{\prime}{-} s] - s[v \overset{\prime}{-} t]} = \frac{t^2[x \overset{\prime}{-} y]}{t[x \overset{\prime}{-} s] - s[x \overset{\prime}{-} t]}$ by Chapter 2. Article 8.

But $\frac{t[x \overset{\prime}{-} s] - s[x \overset{\prime}{-} t]}{t^2}$ is $= \left[x \overset{\prime}{-} \frac{s}{t}\right] = \left[x \overset{\prime}{-} \frac{1}{[x \overset{\prime}{-} z]}\right] = -\frac{[x \overset{\prime\prime}{-} z]}{[x \overset{\prime}{-} z]^2} = -\frac{[x \overset{\prime}{-} y] \times [x \overset{\prime\prime}{-} y]}{[x \overset{\prime}{-} z]^3}$. Therefore, by substitution, we have

$$\mathrm{R} = -\frac{[x \overset{\prime}{-} y] \times [x \overset{\prime}{-} z]^2}{[x \overset{\prime\prime}{-} z]} = -\frac{[x \overset{\prime}{-} z]^3}{[x \overset{\prime\prime}{-} y]},$$

where $[x \overset{\prime}{-} z]$ is $= \sqrt{1 + [x \overset{\prime}{-} y]^2}$.

From

From thoſe equations, and what is proved in Chap. 2. Art. 9. other expreſſions for the value of R may be obtained, viz.

$$R = \frac{[y \dot{-} x] \times [y \dot{-} z]^2}{[y \ddot{-} z]} = \frac{[y \dot{-} z]^3}{[y \ddot{-} x]};$$

$$\text{and } R = \frac{[z \dot{-} y]}{[z \ddot{-} x]} = -\frac{[z \dot{-} x]}{[z \ddot{-} y]}.$$

Or the ſame may be obtained from theſe analogies, viz. $y : t :: u + b - x : R :: [v \dot{-} u] : [v \dot{-} R]\ (= [v \dot{-} w])$, which follow from ſimilar triangles and Chap. 5. Art. 5.

Example. *In the Parabola,* ax being $= y^2$, x is $= \frac{y^2}{a}$, $[y \dot{-} x] = \frac{2y}{a}$, $[y \ddot{-} x] = \frac{2}{a}$, and $[y \dot{-} z]\ (= \sqrt{1 + [y \dot{-} x]^2})$ $= \sqrt{1 + \frac{4y^2}{a^2}}$. Therefore $R = \frac{[y \dot{-} z]^3}{[y \ddot{-} x]}$ is, in ſuch curve, equal to $\frac{\overline{a^2 + 4y^2}|^{\frac{3}{2}}}{2a^2}$.

In the Ellipſis and Hyperbola $2amx \mp ax^2$ is $= 2my^2$: and by reſidual diviſion, and ſubſtitution,

$$R \text{ is found} = \frac{\overline{a^2m^2 + 4m^2y^2 \mp 2amy^2}|^{\frac{3}{2}}}{2a^2m^3}:$$

a being the Parameter, and m the ſemi-tranſverſe Axis.

It is obſervable, that, in each of the Conic Sections, the Radius of Curvature, at the Vertex of the Figure, is equal to *Half the Parameter.*

5.

COROLLARY.

The equation of the involute curve being given, we may, from thence and what is done above, readily find the nature of the evolute. For, from what is ſaid in the laſt article, we have

$$v = \frac{sR}{t} - y - c = \frac{R}{[x \dot{-} z]} - y - c \begin{cases} = -\dfrac{[x \dot{-} z]^2}{[x \ddot{-} y]} - y - c, \\ = \dfrac{[y \dot{-} x] \times [y \dot{-} z]^2}{[y \ddot{-} x]} - y - c; \end{cases}$$

$u =$

$$u=\frac{yR}{t}+x-b=\frac{R}{[y\perp z]}+x-b\begin{cases}=\dfrac{[y\perp z]^2}{[y\perp\!\!\!\perp x]}+x-b,\\[2ex] =-\dfrac{[x\perp y]\times[x\perp z]^2}{[x\perp\!\!\!\perp y]}+x-b.\end{cases}$$

From whence, by means of the given equation, x and y may be expunged: Consequently the relation between v and u will then appear.

EXAMPLE. *Let the involute be the conical Parabola, whose equation is $ax=y^2$.*

Then $[y\perp x]$ being $=\frac{2y}{a}$, $[y\perp\!\!\!\perp x]=\frac{2}{a}$, and $[y\perp z]^2=1+\frac{4y^2}{a^2}$;

$$v \text{ is } =\frac{4y^3}{a^2}-c, \text{ and } u=\frac{a}{2}+\frac{3y^2}{a}-b.$$

$$\text{Hence } y=\overline{\frac{a^2v+a^2c}{4}}\Big|^{\frac{1}{3}}=\overline{\frac{au-\frac{1}{2}a^2+ab}{3}}\Big|^{\frac{1}{2}}.$$

$$\text{Suppose } c=0, \text{ and } b=\tfrac{1}{2}a; \text{ then } \overline{\frac{a^2v}{4}}\Big|^{\frac{1}{3}}=\overline{\frac{au}{3}}\Big|^{\frac{1}{2}},$$

$$\text{and consequently } \frac{27av^2}{16}=u^3.$$

Therefore the evolute aCd is, in this example, the semicubical Parabola: And AI being $=\frac{a}{2}$, and G coinciding with the point I, that point is the center of curvature corresponding to the vertex A of the Parabola APD.

6.

The curve APD being of the spiral kind, whose ordinates all issue from the point G, suppose aCd to be the curve by whose evolution APD is described: join CG, CP; and draw GH, GQ perpendicular to CP and the tangent PQ respectively. Fig. 32.

Call GP, y; GQ, ($=$ HP,) p; the curve aCd, w; and (CP) the radius of curvature at P, R.

Then, CH being $=R-p$, and GH $=\sqrt{y^2-p^2}$, GC will be $=\sqrt{R^2-2pR+y^2}$. Now, by Corollary to Article 8. Chap. 5.

CH

$$CH : CG :: [u \mathbin{\dot{-}} \sqrt{R^2 - 2pR + y^2}] : [u \mathbin{\dot{-}} w] (= [u \mathbin{\dot{-}} R])$$

$$\text{i. e. } R - p : \sqrt{R^2 - 2pR + y^2}$$

$$:: \frac{R[u \mathbin{\dot{-}} R] - p[u \mathbin{\dot{-}} R] - R[u \mathbin{\dot{-}} p] + y[u \mathbin{\dot{-}} y]}{\sqrt{R^2 - 2pR + y^2}} : [u \mathbin{\dot{-}} R].$$

From whence it is evident, that

$$y[u \mathbin{\dot{-}} y] - R[u \mathbin{\dot{-}} p] \text{ is } = 0;$$

and consequently $R = \frac{y[u \mathbin{\dot{-}} y]}{[u \mathbin{\dot{-}} p]}$, u being any function of y.

Example. *Suppose* APD *to be the Logarithmic Spiral; and* s *the sine of the invariable angle* GPQ, *made by the ray* GP *and tangent* PQ, *radius being unity.*

Then GQ ($=p$) will be $= sy$, and consequently $R = \frac{y[u \mathbin{\dot{-}} y]}{[u \mathbin{\dot{-}} p]} = \frac{y}{s}$.

Corollary. CP ($=$ R) being $= \frac{y}{s}$, and $p = sy$; GH will be $= \sqrt{y^2 - s^2y^2} = \sqrt{1 - s^2} \times y$, and CG $= \sqrt{\frac{y^2}{s^2} - y^2} = \sqrt{1 - s^2} \times \frac{y}{s}$. Therefore the sine of the angle GCH will always be $= s$; and consequently, in this example, the evolute aCd is the same curve as the involute APD, but placed in a different position.

7.

Fig. 33. Suppose cpq to be a kind of cycloid described by the point p carried about with the curve aB revolving along the immoveable circular arc AB, as in Art. 10. Chap. 5.

From D, the center of the immoveable circular arc, draw Dg perpendicular to pB continued; and, to the continuation of DB, draw the perpendicular pc. Call the radius DB, R; Bp, x; Dp, y; Bg, w; and let fp, the radius of curvature of the cycloid at p, be called E. Then the tangent pt being (by Chap. 5. Art. 10.) parallel to gD, Dt ($= p$) the perpendicular from D on that tangent will be $= w + x$. Moreover, R being to

to w as x to Bc ($= \frac{wx}{R}$), $\overline{cp}|^2$ will be $= x^2 - \frac{w^2x^2}{R^2}$, and

$$y^2 = \overline{cD}|^2 + \overline{cp}|^2 = \overline{R + \frac{wx}{R}}|^2 + x^2 - \frac{w^2x^2}{R^2} = R^2 + x^2 + 2wx.$$

Therefore E ($= \frac{y[u \dot{-} y]}{[u \dot{-} p]}$ by the last Article) is equal to $\frac{x[u \dot{-} x] + w[u \dot{-} x] + x[u \dot{-} w]}{[u \dot{-} w] + [u \dot{-} x]}$.

SCHOLIUM. If R be infinite, i. e. if AB (instead of being a curve) be a right line: Then Ag being perpendicular on pB; and Bg, Bp, Ap, and AB being called w, x, y, and z respectively; we have p (the perpendicular from A on the tangent to the cycloid at the point p) $= x - w$, and $y^2 = \overline{Ag}|^2 + \overline{pg}|^2 = z^2 - w^2 + \overline{x - w}|^2 = z^2 + x^2 - 2wx$. Therefore, in this case, E is $= \frac{[u \dot{-} \overline{z^2 + x^2 - 2wx}]}{2.[u \dot{-} \overline{x - w}]}$. Fig. 34.

EXAMPLE I. *Let* aB *be a circular arc whose center is* d; *and call the radius* Bd, r; dp, ϱ. Then, $x^2 - \overline{Bc}|^2$ being $= \overline{cp}|^2 = \varrho^2 - \overline{Bc - r}|^2 = \varrho^2 - r^2 + 2r \times Bc - \overline{Bc}|^2$, it is evident that B$c$, or its equal $\frac{wx}{R}$, is $= \frac{x^2 + r^2 - \varrho^2}{2r}$: Hence $w = \frac{R}{2rx} \times \overline{x^2 + r^2 - \varrho^2}$. Fig. 33.

Consequently p ($= w + x$) is $= \frac{R + 2r}{2r} . x + \frac{R . \overline{r^2 - \varrho^2}}{2r} . x^{-1}$,

$$y^2 = R^2 + \frac{R}{r} . \overline{r^2 - \varrho^2} + \frac{R + r}{r} . x^2, \text{ and}$$

$$E = \frac{2 . \overline{R + r} . x^3}{\overline{R + 2r} . x^2 - R . \overline{r^2 - \varrho^2}} = \frac{\overline{R + r} . x^2}{\overline{R + r} . x - rw}.$$

COROLLARY I. Suppose the describing point p to be in the periphery of the moveable circle. Then, ϱ being $= r$, p is $= \frac{R + 2r}{2r} . x$, $y^2 = R^2 + \frac{R + r}{r} . x^2$, and $E = \frac{2 . \overline{R + r}}{R + 2r} . x$.

From whence we have $y^2 = R^2 + p^2 - \frac{R^2}{\overline{R+2r}|^2}\cdot p^2$, and $\sqrt{E^2 - 2pE + y^2} = Df = \overset{''}{y}$ (the ray from D to the point of the evolute correſponding to the point p of the involute) $= \sqrt{R^2 - \frac{R^2.\overline{R+r}}{r.\overline{R+2r}|^2}.x^2}$. Now $Dg = \overset{''}{p}$ being $= \sqrt{R^2 - w^2} = \sqrt{R^2 - \frac{R^2x^2}{4r^2}}$, x^2 is $= 4r^2 - \frac{4r^2\overset{''}{p}^2}{R^2}$, and conſequently $\overset{''}{y}^2 = \frac{R^4}{\overline{R+2r}|^2} + \overset{''}{p}^2 - \frac{R^2}{\overline{R+2r}|^2}.\overset{''}{p}^2$.

Bringing the laſt expreſſion to the ſame form with the value of y^2 we have $\overset{''}{y}^2 = \overset{''}{R}^2 + \overset{''}{p}^2 - \frac{\overset{''}{R}^2}{\overline{\overset{''}{R}+2\overset{''}{r}}|^2}.\overset{''}{p}^2$, where $\overset{''}{R}^2$ muſt be $= \frac{R^4}{\overline{R+2r}|^2}$ and $\frac{\overset{''}{R}^2}{\overline{\overset{''}{R}+2\overset{''}{r}}|^2} = \frac{R^2}{\overline{R+2r}|^2}$. Therefore, $\overset{''}{R}$ is $= \frac{R^2}{R+2r}$, $\overset{''}{r} = \frac{Rr}{R+2r}$, and $\overset{''}{R} + 2\overset{''}{r} = R$. Hence it is plain, that the evolute of the cycloid *opq* is another cycloid, which may be deſcribed by a point in the periphery of a circle whoſe radius is $\frac{Rr}{R+2r}$ revolving upon the immoveable circle whoſe radius is $\frac{R^2}{R+2r}$ and center D; and the poſition of the evolute with reſpect to the involute is very obvious.

COROLLARY II. Let the fixed point P be ſo ſituated, that AP and DP ſhall be reſpectively equal to a*p*, *dp*; and let r be ſuppoſed equal to R. Then, the arcs AB, aB being equal, if the ray PB be drawn, the angle DBP will always be equal to the angle DB*g*. Therefore the evolute of the cycloid *opq* is the cauſtic by reflection of the circle AB, P being the focus of the incident rays.

Now, R being $= r$, E will be $= \frac{4x^3}{3x^2 - r^2 + \varrho^2} = \frac{2x^2}{2x - w}$, and B$f$ = E — x (the diſtance of the focus of the reflected rays from the

the point of incidence) $= \frac{x^3 + \overline{r^2 - \varrho^2} \cdot x}{3x^2 - r^2 + \varrho^2} = \frac{wx}{2x - w}$, where x is $=$ PB, and $\varrho =$ PD.

Hence it is obvious, that if ϱ be $= r$, i. e. if the focus of the incident rays be any where in the periphery of the circle AB, the focal diſtance Bf will always be $= \frac{x}{3} = \frac{2w}{3}$.

COROLLARY III. Let r and ϱ be ſuppoſed each equal to $\frac{1}{2}$R. Then, the arcs AB, aB being equal, and the deſcribing point p coinciding with a, if any ray PB be drawn perpendicular to AD, the angles DBP, DBf will always be equal. Fig. 35.

Therefore the evolute of the cycloid opq is the cauſtic by reflection of the circle AB, the incident rays being parallel.

Now, $\frac{1}{2}$R being $= r = \varrho$, E will be $= \frac{3}{2}x$; and the focal diſtance B$f = \frac{1}{2}x = \frac{1}{2}w$.

COROLLARY IV. AB being any reflecting curve whatever; the focus of rays reflected from any point (B) thereof may be found by the above theorems, by conſidering R as the radius, and D as the center of curvature of ſuch curve at the point of incidence.

EXAMPLE II. *Let* ap *be perpendicular to the tangent to the moveable curve at* a, *and the points* P, A, D *in a right line: and ſuppoſe the curve* aB *of ſuch a nature, that, the arc* aB *being equal to the arc* AB, pB $(= x)$ *ſhall always be equal to* n $\times$ PB, n *being invariable.* Fig. 36.

Then PB being continued, and Dh drawn perpendicular thereto, $n\sqrt{R^2 - v^2}$ will be $= \sqrt{R^2 - w^2}$, v being put for Bh: for it follows from Chap. 5. Art. 8. that the ſines of the angles DBh, DBg will be to each other in the invariable ratio of 1 to n reſpectively; thoſe ſines being reſpectively equal to the coſine of the angle made by PB and the common tangent to the two curves at B, and the coſine of the angle made by pB and the ſame tangent.—Moreover, calling AP, d; $\overline{d + R}|^2 - R^2 + v^2$ will be $= \overline{v + \frac{x}{n}}|^2$.—By means of which equations

$[u \dot{-} w]$ and $[u \dot{-} x]$ may be eaſily expunged out of the equation $E = \frac{x [u \dot{-} x] + w [u \dot{-} x] + x [u \dot{-} w]}{[u \dot{-} w] + [u \dot{-} x]}$; and then we ſhall have

$$E = \frac{\overline{w - nv} \,.\, x^2 + \overline{w^2 - n^2v^2} \,.\, x}{\overline{w - nv} \,.\, x - n^2v^2}.$$

Corollary I. The ſines of the angles DBh, DBg being to each other in the invariable ratio of 1 to n reſpectively, the evolute of the curve cpq is the cauſtic by refraction of the circle AB, P being the focus of the incident rays, and 1 to n the ratio of the ſine of incidence to that of refraction. Conſequently, taking x from the value of E, and writing ny inſtead of x; Bf, the diſtance of the focus of the refracted rays from the point of incidence, will be found $= \frac{w^2y}{wy - nvy - nv^2}$, where y is equal to PB.

Corollary II. AB being any refracting curve whatever, the focus of rays refracted at any point (B) thereof may be found by the theorem in the preceding Corollary, by conſidering R as the radius, and D as the center of curvature of ſuch curve at the point of incidence.

Fig. 34. Scholium. If R be infinite, i. e. if AB (inſtead of being a curve) be a right line, to which AP is perpendicular. Then, Ah, Ag being reſpectively perpendicular to BP, Bp; and AP, Bh, Bg, Bp, BP, and AB being called d, v, w, x, y, and z reſpectively; we have $y^2 - d^2 = z^2 = vy$, $x = ny$, and $w = nv$. From whence we get $x - w = ny - nv = ny - \frac{ny^2 - nd^2}{y} = \frac{nd^2}{y}$; and $z^2 + x^2 - 2wx = y^2 - d^2 + n^2y^2 - 2n^2vy = y^2 - n^2y^2 - d^2 + n^2d^2$.

Therefore, by the Scholium to Article 7. E is $= \frac{\overline{n^2 - 1} \,.\, y^3}{nd^2}$.

Conſequently, in this caſe, by what is obſerved in Corollary 1. $E - x$, the diſtance of the focus of the refracted rays from the point of incidence B, is $= \frac{\overline{n^2 - 1} \,.\, y^3}{nd^2} - ny$, P being the focus of the incident rays.

8. To

8.

To find the point F where any refracted ray BF meets the axis AF of any refracting curve AB. Draw the incident ray PB, the ordinate BC, the tangent Bt, and BD perpendicular to that tangent. Call the abſciſſa AC, x; the ordinate BC, y; the curve AB, z; PB, v; BF, w; AP, d; AF, e: and let the ſine of incidence be to that of refraction, as 1 to n. Then, the ſubtangent, ordinate, and tangent being to each other as 1, $[x \dot{-} y]$, and $[x \dot{-} z]$ reſpectively, by Chap. 5. Art. 5. we have Fig. 37.

$1 : [x \dot{-} y] :: y : y[x \dot{-} y] = CD$; and (radius being 1)

$[x \dot{-} z] : 1 :: 1 : \frac{1}{[x \dot{-} z]}$, the ſine of the angle tBC = BDC;

$[x \dot{-} z] : 1 :: [x \dot{-} y] : \frac{[x \dot{-} y]}{[x \dot{-} z]}$, the coſine of the ſame angle.

Moreover $v : 1 :: d+x : \frac{d+x}{v}$, the ſine of the angle PBC;

and $v : 1 :: \quad y \quad : \frac{y}{v}$, the coſine of PBC.

Now the coſine of the difference of two angles being equal to the rectangle of the two ſines *plus* the rectangle of the two coſines of thoſe two angles, $\frac{d+x}{v[x \dot{-} z]} + \frac{y[x \dot{-} y]}{v[x \dot{-} z]}$ is the coſine of the angle tBP, or the ſine of the angle of incidence.

Therefore, n times this laſt ſine being the ſine of the angle of refraction DBF, we have

$$n \times \frac{d+x+y[x \dot{-} y]}{v[x \dot{-} z]} : \frac{1}{[x \dot{-} z]} :: e - x - y[x \dot{-} y]\ (= DF) : w.$$

Conſequently, v being $= \sqrt{y^2 + \overline{d+x}|^2}$, and $w = \sqrt{y^2 + \overline{e-x}|^2}$,

$$n \times \frac{d+x+y[x \dot{-} y]}{\sqrt{\overline{d+x}|^2 + y^2}} \text{ is } = \frac{e-x-y[x \dot{-} y]}{\sqrt{\overline{e-x}|^2 + y^2}};$$

by means of which equation, and the equation of the given refracting curve, e may be readily determined.

SCHOLIUM I. If d be infinite, i. e. if the incident ray $\overset{\prime\prime}{P}$B be parallel to the axis of the curve, the ſine of $\overset{\prime\prime}{P}$BC will be $= 1$,

and

and the cofine $= 0$. Therefore, in that cafe, $n : 1 :: \mathrm{DF} : \mathrm{BF}$, and confequently $n = \dfrac{e - x - y[x \perp y]}{\sqrt{\overline{e - x}|^2 + y^2}}$.

Scholium II. That the refracted rays may be all parallel, the angle made by BD and any ray after refraction at B muft be equal to the angle BDC. Therefore $n \times \dfrac{d + x + y[x \perp y]}{v[x \perp z]}$ muft be $= \dfrac{1}{[x \perp z]}$; and confequently, in this cafe, $n \times \dfrac{d + x + y[x \perp y]}{\sqrt{\overline{d + x}|^2 + y^2}}$ muft be $= 1$.

Example I. *The refracting curve being* Descartes' *Oval, whofe equation is* $n\sqrt{\overline{a + x}|^2 + y^2} - na = b - \sqrt{\overline{b - x}|^2 + y^2}$; we, by refidual divifion, get $n \times \dfrac{a + x + y[x \perp y]}{\sqrt{\overline{a + x}|^2 + y^2}} = \dfrac{b - x - y[x \perp y]}{\sqrt{\overline{b - x}|^2 + y^2}}$.

Which equation being compared with the equation $n \times \dfrac{d + x + y[x \perp y]}{\sqrt{\overline{d + x}|^2 + y^2}} = \dfrac{e - x - y[x \perp y]}{\sqrt{\overline{e - x}|^2 + y^2}}$ above found, it appears that the fines of incidence and refraction being as 1 to n, if (n being lefs than 1, and a and b pofitive quantities,) d be $= a$, e will always be $= b$; and the refracted rays, from every point of the curve, will all meet the axis in one and the fame point, which therefore will be their common focus, without any aberration.

Example II. *Suppofe the curve* AB *to be* a circle, *whofe radius is* a. Then, y^2 being $= 2ax - x^2$, $y[x \perp y]$ is $= a - x$; and by fubftituting thefe values of y^2 and $y[x \perp y]$ for their refpective equals in the equation $n \times \dfrac{d + x + y[x \perp y]}{\sqrt{\overline{d + x}|^2 + y^2}} = \dfrac{e - x - y[x \perp y]}{\sqrt{\overline{e - x}|^2 + y^2}}$, we have $\dfrac{n . \overline{a + d}}{\sqrt{d^2 + 2 . \overline{a + d} . x}} = \dfrac{e - a}{\sqrt{e^2 + 2 . \overline{a - e} . x}}$: from whence, when a, d, n, and x are given, e may be readily found; and, confequently, the aberration of any ray from the focus of rays falling on or very near the vertex A.

From

From the laſt equation we get $\frac{\sqrt{d^2 + 2 . \overline{a+d} . x}}{\sqrt{e^2 + 2 . \overline{a-e} . x}} = \frac{n . \overline{a+d}}{e-a}$; therefore $\frac{\sqrt{b^2d^2 + 2b^2 . \overline{a+d} . x}}{\sqrt{e^2 + 2 . \overline{a-e} . x}}$ is $= \frac{bn . \overline{a+d}}{e-a}$, b being of any value whatever. Now ſuppoſing $e^2 = b^2d^2$, and $a - e = b^2 \times \overline{a+d}$, $\frac{bn . \overline{a+d}}{e-a}$, it is evident, muſt be $= 1$, let x be what it will: From which equations we have $e = bd = -nd = \overline{n+1} . a$, and $d = -\frac{n+1}{n}a$. Hence it appears, that if P be on the concave ſide of AB, and its diſtance from A be $= \frac{n+1}{n}a$, the refracted rays from every point of the curve will all diverge from one point in the axis, the diſtance of which point from A will be $= \overline{n+1} . a$.

And it follows, that rays converging to P, (ſituated as juſt now mentioned,) and falling on the convexity of the circle, will all be refracted to F, AF being $= \overline{n+1} . a$.

EXAMPLE III. *Let* AB *be* an ellipſis, *whoſe equation is* $y^2 = ax - \frac{a}{b}x^2$, *a being the parameter, and b the tranſverſe axis.*

Then will $-\sqrt{\overline{g-x}|^2 + y^2}$ be $= -\sqrt{\overline{g-x}|^2 + ax - \frac{a}{b}x^2}$, g being of any value whatever. Hence, by reſidual diviſion, we get $\frac{g - x - y [x \perp y]}{\sqrt{\overline{g-x}|^2 + y^2}} = \frac{g - x - \frac{1}{2}a + \frac{a}{b}x}{\sqrt{\overline{g-x}|^2 + ax - \frac{a}{b}x^2}}$. By comparing this equation with that in Schol. 1. it appears that, the incident rays being parallel to the axis of the curve, e will always be equal to the invariable quantity g, let x be what it will, if $\frac{g - x - \frac{1}{2}a + \frac{a}{b}x}{\sqrt{\overline{g-x}|^2 + ax - \frac{a}{b}x^2}}$ be always $= n$, or $\overline{g - \frac{1}{2}a + \frac{a}{b} - 1 . x}|^2$ always $= n^2 \times \overline{g-x}|^2 + ax - \frac{a}{b}x^2$; i. e. $\overline{g - \frac{1}{2}a}|^2 +$

2 .

$2 \cdot \overline{g - \frac{1}{2}a} \cdot \overline{\frac{a}{b} - 1} \cdot x + \overline{\frac{a}{b} - 1}\rvert^2 \cdot x^2$ always $= n^2 g^2 + n^2 \cdot \overline{a - 2g} \cdot x - n^2 \cdot \overline{\frac{a}{b} - 1} \cdot x^2$. Now, that this laſt may be a true equation, x being of any value whatever, $\overline{g - \frac{1}{2}a}\rvert^2$ muſt be $= n^2 g^2$, $2 \cdot \overline{g - \frac{1}{2}a} \cdot \overline{\frac{a}{b} - 1} = n^2 \cdot \overline{a - 2g}$ $(= - 2n^2 \cdot \overline{g - \frac{1}{2}a})$, and $\overline{\frac{a}{b} - 1}\rvert^2 = - n^2 \cdot \overline{\frac{a}{b} - 1}$. And from theſe equations we have $\frac{a}{b} = 1 - n^2$, and $g = \frac{1}{2}a \times \frac{1}{1 - n} = \frac{1}{2}b \times \overline{1 + n}$; which value of g is the diſtance of the remoter focus of the ellipſis from the vertex, when $\frac{a}{b}$ is $= 1 - n^2$. It is evident therefore, that, $\frac{a}{b}$ being $= 1 - n^2$, and the incident rays being parallel to the tranſverſe axis, and falling on the convexity of the curve, the refracted rays from every point of the figure will, without any aberration, all converge to the focus juſt now mentioned.

And (reverſing the ratio of the ſines of incidence and refraction) it is obvious that, $\frac{a}{b}$ being $= \frac{n^2 - 1}{n^2}$, and the incident rays iſſuing from one of the focuſes of the ellipſis and falling on the concavity of the farther half thereof, the rays after refraction will all proceed in a direction parallel to the tranſverſe axis.

Writing $-b$ inſtead of b, the equation of the curve becomes $y^2 = ax + \frac{a}{b}x^2$, correſponding to an hyperbola whoſe parameter is a, and tranſverſe axis b. It follows therefore, that, $\frac{a}{b}$ being $= n^2 - 1$, and the incident rays being parallel to the tranverſe axis, and falling on the concavity of the curve AB, now ſuppoſed an hyperbola, the refracted rays will all converge to the focus of the conjugate hyperbola,

And (reverſing the ratio of the ſines of incidence and refraction) it is obvious that, $\frac{a}{b}$ being $= \frac{1 - n^2}{n^2}$, and the incident rays iſſuing from the focus of the conjugate hyperbola, and falling on the convexity of the hyperbola AB, the rays, after refraction at the laſt-mentioned curve, will, from every point thereof, proceed in a direction parallel to the tranſverſe axis.

THE

THE

RESIDUAL ANALYSIS.

CHAP. VIII.

Of the greateſt *and* leaſt ORDINATES, *the* POINTS *of* contrary flexion *and* reflexion, *and the* double *and* triple &c. POINTS *of* curve Lines.

I.

IF, y being ſome function of x, $y_{,}$ be always leſs or always greater than y, when $x_{,}$ taken between certain limits, is either leſs or greater than x, how near ſoever $x_{,}$ be taken to x; the value of y, or the quantity denoted thereby, is conſidered as a *maximum* or *minimum*, without regard to any value it may have when x is not taken between the ſaid limits.

2.

The ordinate from a point of a curve is a *maximum*, or *minimum*, when, the curve being immediately continued on both ſides thereof, it is greater, or leſs, than the ordinates which may be drawn, on either ſide of it, from the adjoining parts of the curve.

3.

Suppose y to be the ordinate of a curve corresponding to the abscissa x; and $y_{\prime}$ another ordinate (of the same curve) corresponding to the abscissa $x_{\prime}$. Then if y be a *maximum*, it will be greater than $y_{\prime}$, whether $x_{\prime}$ be less or greater than x, provided $x_{\prime}$ be taken between certain limits: and consequently the residual $y - y_{\prime}$ will, in such case, be always positive, when $x_{\prime}$ is taken either less or greater than x, between those limits.

But if y be a *minimum*, $y - y_{\prime}$ will be always negative, when $x_{\prime}$ is taken either less or greater than x, between certain limits.

Now seeing that, when y is a *maximum* or *minimum*, the value of the residual $y - y_{\prime}$ must be always positive or always negative, when $x_{\prime}$ is taken either less or greater than x, between certain limits: it is obvious, that, in such case, the value of $\overline{y - y_{\prime}} \div \overline{x - x_{\prime}}$ will, accordingly, be positive when $x_{\prime}$ is less than x, and negative when $x_{\prime}$ is greater than x; or positive when $x_{\prime}$ is greater than x, and negative when $x_{\prime}$ is less than x, how near soever $x_{\prime}$ be taken to x. Therefore, by Chap. 4. Art. 1.

$$[x \dot{\perp} y] \text{ will then be } = 0, \text{ or } \frac{1}{[x \dot{\perp} y]} = 0.$$

Moreover, supposing b to denote a value of x in either of those equations, and d the correspondent value of y, d being a *maximum* or a *minimum*, $\frac{y-d}{x-b}$ will, accordingly, be positive when x is less than b, and negative when x is greater than b; or positive when x is greater than b, and negative when x is less than b, how

how near ſoever x be taken to b, between certain limits. Therefore, by what is ſaid in Chap. 4. Art. 3. that expreſſion, viz.

$$\frac{y-d}{x-b} \text{ will be } = \overline{x-b}\rvert^{m} \times Q;$$

m being ſuppoſed an odd number, or a fraction whoſe numerator and denominator are both odd numbers; and Q ſome algebraic expreſſion conſiſting of ſuch quantities, that its value ſhall be real when x is either leſs or greater than b, (between certain limits,) and that neither it nor its reciprocal ſhall vaniſh when x is equal to b.

And it likewiſe follows from the article laſt mentioned, that, the abſciſſa being equal to b, the correſpondent ordinate ſhall be a *maximum* or a *minimum*, according as q, the value of Q when x is equal to b, is negative or poſitive, m being as juſt now ſpecified. But, if m comes out contrary to our ſuppoſition, the abſciſſa, when equal to b, will not correſpond to an ordinate that is either a *maximum* or a *minimum*.—This we ſhall, by and by, explain by proper examples.

The equation $\frac{y-d}{x-b} = \overline{x-b}\rvert^{m} \times Q$ being divided by $\overline{x-b}\rvert^{m}$, we have $\frac{y-d}{\overline{x-b}\rvert^{m+1}} = Q$; where ($d$ being finite) $m+1$ muſt be poſitive, otherwiſe q and $\frac{1}{q}$ cannot be finite. Now, by Chap. 2. Cor. to Art. 6. the value of $\frac{y-d}{\overline{x-b}\rvert^{m+1}}$, when x is $= b$, is equal to the value of the quotient of $[x \dot{-} y]$ divided by $\overline{m+1} \cdot \overline{x-b}\rvert^{m}$, when x is taken $= b$: Therefore q will be equal to the value of $\frac{[x \dot{-} y]}{\overline{m+1} \cdot \overline{x-b}\rvert^{m}}$, when x is taken as juſt now mentioned. Hence it is evident, that m and q may be found by reſolving $[x \dot{-} y]$ into two ſuch factors, (F and G,) that one of them (F) ſhall be ſome power of $x-b$, and the other (G) ſhall neither vaniſh nor become infinite when x is therein taken equal to b: for by comparing $\overline{x-b}\rvert^{m}$ with F, m will be known; and q will be the value of $\frac{G}{m+1}$ when x is equal to b. Moreover, Q will be real or imaginary when x is taken leſs

or greater than b, between certain limits; according as G is real or imaginary when x is so taken. Therefore, by means of the said factors F and G, we may readily know whether the ordinate corresponding to the abscissa b be a *maximum* or *minimum*, without assigning the general value of Q.

EXAMPLE I. *Suppose* $y = ax - x^2$, a *being invariable.*

Then we shall have $[x \perp y] = a - 2x = \overline{x - \frac{a}{2}} \times -2 = 0$; where x is $= \frac{a}{2}$.

Now, b being $= \frac{a}{2}$, we have $F = x - \frac{a}{2} = \overline{x - b}|^m$, and $G = -2$. Therefore, m being $= 1$, and q (the value of $\frac{G}{m+1}$ when x is $= b$) being a negative quantity, y is a *maximum* when x is $= \frac{a}{2}$.

EXAMPLE II. *Suppose* $y = x^4 - a^3x$, *where* a *is invariable.*

Then we have

$$[x \perp y] = 4x^3 - a^3 = \overline{x - \frac{a}{4^{\frac{1}{3}}}} \times \overline{4^{\frac{1}{3}}a^2 + 4^{\frac{2}{3}}ax + 4x^2} = 0;$$

where the real value of x is $\frac{a}{4^{\frac{1}{3}}}$.

Now, b being $= \frac{a}{4^{\frac{1}{3}}}$, we have $F = x - \frac{a}{4^{\frac{1}{3}}} = \overline{x - b}|^m$, and $G = 4^{\frac{1}{3}}a^2 + 4^{\frac{2}{3}}ax + 4x^2$. Therefore, m being $= 1$, and q (the value of $\frac{G}{m+1}$ when x is $= b$) being a positive quantity, y is a *minimum* when x is $= \frac{a}{4^{\frac{1}{3}}}$.

EXAMPLE III. *Suppose* $y = x^2 + \overline{a^3 - x^3}|^{\frac{2}{3}}$, *where* a *is an invariable positive quantity.*

Then we shall have

$$[x \perp y] = \frac{2x \times \overline{a^3 - x^3}|^{\frac{1}{3}} - 2x^2}{\overline{a^3 - x^3}|^{\frac{1}{3}}} \left(= \frac{8a^3x - 16x^4}{\overline{a^3 - x^3}|^{\frac{1}{3}} \times \overline{g^2x^{-2} + 6g + 12x^2}}\right.^* =$$

* $2x \times \overline{a^3 - x^3}|^{\frac{1}{3}} - 2x^2$ being supposed $= g$, $2x \times \overline{a^3 - x^3}|^{\frac{1}{3}}$ will be $= g + 2x^2$, and

$$= \frac{-8x \times \overline{x - \frac{a}{2^{\frac{1}{3}}}} \times \overline{2^{\frac{1}{3}}a^2 + 2^{\frac{2}{3}}ax + 2x^2}}{\overline{a^3 - x^3}|^{\frac{1}{3}} \times \overline{g^2x^{-2} + 6g + 12x^2}},$$ g being put for the numerator $2x \times \overline{a^3 - x^3}|^{\frac{1}{3}} - 2x^2) = 0$; and $\frac{1}{[x \mathbin{\dot{-}} y]} =$

$$\frac{\overline{a^3 - x^3}|^{\frac{1}{3}}}{2x \times \overline{a^3 - x^3}|^{\frac{1}{3}} - 2x^2} = \frac{\overline{x - a}|^{\frac{1}{3}} \times \overline{a^2 + ax + x^2}|^{\frac{1}{3}}}{2x^2 - 2x \times \overline{a^3 - x^3}|^{\frac{1}{3}}} = 0.$$

In the equation $[x \mathbin{\dot{-}} y] = o$, the real values of x are o and $\frac{a}{2^{\frac{1}{3}}}$; and, in the equation $\frac{1}{[x \mathbin{\dot{-}} y]} = o$, the real value of x is a.

Taking b equal to o, we have $F = x - o = \overline{x - b}|^m$, and $G = \frac{-8 \times \overline{x - \frac{a}{2^{\frac{1}{3}}}} \times \overline{2^{\frac{1}{3}}a^2 + 2^{\frac{2}{3}}ax + 2x^2}}{\overline{a^3 - x^3}|^{\frac{1}{3}} \times \overline{g^2x^{-2} + 6g + 12x^2}}$. Therefore, m being manifestly $= 1$, and q (the value of $\frac{G}{m + 1}$ when x is $= b$) being a positive quantity, y is a *minimum* when x is $= o$.

Taking b equal to $\frac{a}{2^{\frac{1}{3}}}$, we have $F = x - \frac{a}{2^{\frac{1}{3}}} = \overline{x - b}|^m$, and $G = \frac{-8x \times \overline{2^{\frac{1}{3}}a^2 + 2^{\frac{2}{3}}ax + 2x^2}}{\overline{a^3 - x^3}|^{\frac{1}{3}} \times \overline{g^2x^{-2} + 6g + 12x^2}}$. Therefore, m being $= 1$, and q being a negative quantity, y is a *maximum* when x is $= \frac{a}{2^{\frac{1}{3}}}$.

Taking b equal to a, we have $F = \overline{x - a}|^{-\frac{1}{3}} = \overline{x - b}|^m$, and $G = \frac{2x^2 - 2x \times \overline{a^3 - x^3}|^{\frac{1}{3}}}{\overline{a^2 + ax + x^2}|^{\frac{1}{3}}}$. Therefore m being $= -\frac{1}{3}$, and q being a positive quantity, y is a *minimum* when x is $= a$.

---

and $8a^3x^3 - 16x^6 = g^3 + 6g^2x^2 + 12gx^4$. From whence it is evident, that if g be multiplied by $g^2 + 6gx^2 + 12x^4$, the product will be equal to $8a^3x^3 - 16x^6$; and therefore it is plain that $\frac{2x \times \overline{a^3 - x^3}|^{\frac{1}{3}} - 2x^2}{\overline{a^3 - x^3}|^{\frac{1}{3}}}$ is equal to $\frac{8a^3x^3 - 16x^6}{\overline{a^3 - x^3}|^{\frac{1}{3}} \times \overline{g^2 + 6gx^2 + 12x^4}} = \frac{8a^3x - 16x^4}{\overline{a^3 - x^3}|^{\frac{1}{3}} \times \overline{g^2x^{-2} + 6g + 12x^2}}$, the numerator and denominator being each divided by x^2, the variable quantity by which both $8a^3x^3 - 16x^6$ and $g^2 + 6gx^2 + 12x^4$ are divisible.

EXAMPLE

EXAMPLE IV. *Suppoſe* $y = a^4x - a^3x^2 - \frac{2}{3}a^2x^3 + \frac{3}{2}ax^4 - \frac{3}{5}x^5$, *where* a *is an invariable poſitive quantity.*

Then will $[x \dot{-} y]$ be $= a^4 - 2a^3x - 2a^2x^2 + 6ax^3 - 3x^4 =$

$$-3 \times \overline{x - \frac{a}{\sqrt{3}}} \times \overline{x + \frac{a}{\sqrt{3}}} \times \overline{x - a}\rceil^2 = 0;$$

where x is equal to $\frac{a}{\sqrt{3}}$, or $-\frac{a}{\sqrt{3}}$; or a.

Taking b equal to $\frac{a}{\sqrt{3}}$, we have $F = x - \frac{a}{\sqrt{3}} = \overline{x - b}\rceil^m$, and $G = -3 \times \overline{x + \frac{a}{\sqrt{3}}} \times \overline{x - a}\rceil^2$. Therefore, m being evidently $= 1$, and q (the value of $\frac{G}{m+1}$ when x is $= b$) being a negative quantity, y is a *maximum* when x is $= \frac{a}{\sqrt{3}}$.

Taking b equal to $-\frac{a}{\sqrt{3}}$, we have $F = x + \frac{a}{\sqrt{3}} = \overline{x - b}\rceil^m$, and $G = -3 \times \overline{x - \frac{a}{\sqrt{3}}} \times \overline{x - a}\rceil^2$. Therefore, m being $= 1$, and q being a poſitive quantity, y is a *minimum* when x is $= -\frac{a}{\sqrt{3}}$.

Taking b equal to a, we have $F = \overline{x - a}\rceil^2 = \overline{x - b}\rceil^m$, and $G = -3 \times \overline{x - \frac{a}{\sqrt{3}}} \times \overline{x + \frac{a}{\sqrt{3}}}$. Therefore m is $= 2$; which being contrary to our ſuppoſition, it follows, that y is neither a *maximum* nor a *minimum* when x is $= a$, notwithſtanding this value of x is determined in the ſame manner as are the other two values above ſpecified.

EXAMPLE V. *Suppoſe* $y = x + \overline{a^3 - x^3}\rceil^{\frac{1}{3}}$, *where* a *is an invariable poſitive quantity.*

Then we have $[x \dot{-} y] = \frac{\overline{a^3 - x^3}\rceil^{\frac{2}{3}} - x^2}{\overline{a^3 - x^3}\rceil^{\frac{2}{3}}} (= \frac{a^6 - 2a^3x^3}{\overline{a^3 - x^3}\rceil^{\frac{2}{3}} \times \overline{g^2 + 3gx^2 + 3x^4}}$

$=$

$$= \frac{\overline{x - \frac{a}{2^{\frac{1}{3}}}} \times \overline{-2^{\frac{1}{3}}a^5 - 2^{\frac{2}{3}}a^4x - 2a^3x^2}}{\overline{a^3 - x^3}|^{\frac{2}{3}} \times \overline{g^2 + 3gx^2 + 3x^4}}, \text{ } g \text{ being put for } \overline{a^3 - x^3}|^{\frac{2}{3}} - x^2) = 0;$$

and $\frac{1}{[x \perp y]} = \frac{\overline{a^3 - x^3}|^{\frac{2}{3}}}{\overline{a^3 - x^3}|^{\frac{2}{3}} - x^2} = \frac{\overline{x - a}|^{\frac{2}{3}} \times \overline{a^2 + ax + x^2}|^{\frac{2}{3}}}{\overline{a^3 - x^3}|^{\frac{2}{3}} - x^2} = 0.$

In the equation $[x \perp y] = 0$, the real value of x is $\frac{a}{2^{\frac{1}{3}}}$; and in the equation $\frac{1}{[x \perp y]} = 0$, the real value of x is a.

Taking b equal to $\frac{a}{2^{\frac{1}{3}}}$, we have $F = x - \frac{a}{2^{\frac{1}{3}}} = \overline{x - b}|^m$, and $G = \frac{\overline{-2^{\frac{1}{3}}a^5 - 2^{\frac{2}{3}}a^4x - 2a^3x^2}}{\overline{a^3 - x^3}|^{\frac{2}{3}} \times \overline{g^2 + 3gx^2 + 3x^4}}$. Therefore, m being $= 1$, and q (the value of $\frac{G}{m+1}$ when x is $= b$) being a negative quantity, y is a *maximum* when x is $= \frac{a}{2^{\frac{1}{3}}}$.

Taking b equal to a, we have $F = \overline{x - a}|^{-\frac{2}{3}} = \overline{x - b}|^m$, and $G = \frac{\overline{a^3 - x^3}|^{\frac{2}{3}} - x^2}{\overline{a^2 + ax + x^2}|^{\frac{2}{3}}}$. Therefore m is $= -\frac{2}{3}$; which being contrary to our supposition, it follows, that y is neither a *maximum* nor a *minimum* when x is $= a$.

EXAMPLE VI. *Suppose* $y = a - \frac{\overline{a - x}|^{\frac{5}{2}}}{a^{\frac{3}{2}}} + \frac{\overline{a - x}|^2}{a}$, *where* a *is invariable and positive.*

Then will

$[x \perp y]$ be $= \frac{5 \cdot \overline{a - x}|^{\frac{3}{2}}}{2a^{\frac{3}{2}}} - \frac{2 \cdot \overline{a - x}}{a} = \frac{25 \times \overline{x - \frac{9a}{25}} \times \overline{x - a}}{2a^{\frac{3}{2}} \times \overline{5 \cdot \overline{a - x}|^{\frac{1}{2}} + 4a^{\frac{1}{2}}}} = 0;$

where x is equal to $\frac{9a}{25}$, or to a.

Taking b equal to $\frac{9a}{25}$, we have $F = x - \frac{9a}{25} = \overline{x - b}|^m$, and

and $G = \frac{25 \times \overline{x - a}}{2a^{\frac{3}{2}} \times 5 \cdot \overline{a - x}|^{\frac{1}{2}} + 4a^{\frac{1}{2}}}$. Therefore, m being $= 1$, and q (the value of $\frac{G}{m+1}$ when x is $= b$) being a negative quantity, y is a *maximum* when x is $= \frac{9a}{25}$.

Taking b equal to a, we have $F = x - a = \overline{x - b}|^{m}$, and $G = \frac{25x - 9a}{2a^{\frac{3}{2}} \times 5 \cdot \overline{a - x}|^{\frac{1}{2}} + 4a^{\frac{1}{2}}}$. Here m, which is manifeſtly $= 1$, is agreeable to our ſuppoſition; but, the value of G being imaginary when x is greater than a, it appears, (as it likewiſe does from the equation of the curve,) that, when x is equal to a, the curve is not continued on both ſides of the ordinate; therefore y is not then a *maximum* or *minimum*, within the meaning of our explication in Art. 1. and 2.

It may be worth while to enquire concerning the point of the curve to which the ordinate d (determined as above) correſponds, when m comes out contrary to our ſuppoſition, or G is not real, both when x is greater and when leſs than b: and that I purpoſe to do, in the next article.

4.

Fig. 38. We have found, in Chap. 5. that $\frac{y}{[x \acute{-} y]}$ is the general value
39.
40. of the ſubtangent; therefore $[x \acute{-} y]$ is the quotient of the
41. ordinate divided by the ſubtangent. Now it is obvious, that the
42. 43. ſaid quotient will vaniſh, when, the ordinate being finite, the
44. 45. tangent is parallel to the baſe; and that $\left(\frac{1}{[x \acute{-} y]}\right)$ the reciprocal of
46. the ſame quotient will vaniſh when the tangent coincides with
47. 48. the ordinate. When the tangent is parallel to the baſe, the
49. ordinate may be a *maximum* or *minimum*, as in Fig. 38, 39. or it may paſs through a point of contrary flexion, as in Fig. 40. or correſpond to a cuſpid, as in Fig. 41, 42, 43. Alſo, when the tangent coincides with the ordinate, ſuch ordinate may be a *maximum* or *minimum*, as in Fig. 44, 45. or it may meet the curve in a point of contrary flexure, as in Fig. 46. or correſpond to

to a cuſpid, as in Fig. 47, 48. or touch a continued arch, as in Fig. 49. It appears therefore, that, b being a value of x in the equation $[x \dot{-} y] = o$, or $\frac{1}{[x \dot{-} y]} = o$, and d the correſpondent value of y, (as ſuppoſed above,) the ordinate d may be a *maximum* or *minimum*, or correſpond to a point of contrary flexion or reflexion, or touch a continued arch: moreover, the tangent to the correſpondent point of the curve will be parallel to the baſe, or coincide with the ordinate, according as b is determined from the equation $[x \dot{-} y] = o$, or $\frac{1}{[x \dot{-} y]} = o$; i. e. according as m, in the equation $y - d = \overline{x - b}|^{m+1} \times Q$, is poſitive or negative.

If the ordinate d correſponds to a point of contrary flexure, $\overline{x - b}|^{m+1} \times Q$, the value of $y - d$, muſt be negative when x is leſs than b, and poſitive when x is greater than b; or negative when x is greater than b, and poſitive when x is leſs than b. Therefore $\overline{x - b}|^{m+1}$ muſt, in that caſe, be negative or poſitive, according as x is leſs or greater than b; and conſequently $m + 1$ muſt be an odd number, or a fraction whoſe numerator and denominator are both odd numbers. From whence it follows, that m muſt then be an even number, or a fraction whoſe numerator is an even number and denominator an odd number. It is evident therefore, that, when m is as juſt now ſpecified, and G is real both when x is greater and when leſs than b, the ordinate d meets the curve in a point of contrary flexure; and the tangent at that point will be parallel to the baſe, or coincide with the ordinate, according as m is poſitive or negative: moreover, if x be increaſed after being equal to b, y will, at the ſame time, increaſe or decreaſe, according as q is poſitive or negative.

The celebrated Marquis DE L'HOSPITAL *diſtinguiſhes cuſpids (or points of reflexion,) into two kinds: when the branches of the curve which form the cuſpid have their convexity towards each other, the cuſpid is ſaid to be of the* firſt kind; *but of the* ſecond kind, *when the convexity of one of thoſe branches is towards the concavity of the other.*—In what follows, we will obſerve the ſame diſtinction.

Fig. 50. Let AE be the baſe upon which the abſciſſa x is meaſured when y is conſidered as the ordinate, and let AF be drawn parallel to the ſaid ordinate. Then, y being ſuppoſed to denote any abſciſſa AD, meaſured on AF; and x the correſponding ordinate DP; we have, by the preceding article, $[y \perp x] = o$, or $\frac{1}{[y \perp x]} = o$, when x is a *maximum* or *minimum*. Therefore, $[x \perp y]$ (by Art. 9. Chap. 2.) being $= \frac{1}{[y \perp x]}$, $\frac{1}{[x \perp y]}$ will be $= o$, or $[x \perp y] = o$, when x is as juſt now mentioned. Moreover, from the equation $\frac{y-d}{x-b} = \overline{x-b}|^m \times Q$, we get $\frac{x-b}{y-d} = \overline{y-d}|^{\frac{-m}{m+1}} \times \frac{1}{Q^{\frac{1}{m+1}}}$. Therefore, by the laſt article, $\frac{-m}{m+1}$ muſt be an odd number, or a fraction whoſe numerator and denominator are both odd numbers, when x is a *maximum* or *minimum*: Hence it follows, that m muſt then be a fraction whoſe numerator is an odd number, and denominator an even number.—Now, when DP ($= x$) is a *maximum* or *minimum*, CP (parallel and equal to AD $= y$) either touches a continued arch; or correſponds to a cuſpid of the firſt kind, with the tangent at the point of reflexion parallel to the baſe.—It is manifeſt therefore, that, if, when DP is equal to b, the curve be continued on both ſides of that line, m muſt be as juſt now ſpecified; and the ordinate d will correſpond to ſuch a cuſpid as we laſt mentioned, or touch a continued arch, according as m is poſitive or negative.

From what has been ſaid it is evident, that, m being any number or fraction whatever, if the curve be not continued on both ſides of the ordinate d, that ordinate will correſpond to a cuſpid of the ſecond kind; except when m is a fraction whoſe numerator is an odd number, and denominator an even number, in which caſe the ſaid ordinate may correſpond to a cuſpid of either kind, or touch a continued arch.

Corollary I. The value of $[x \perp y]$ being expreſſed by a fraction, if, by ſuppoſing the numerator thereof $= o$, x be found equal to (any quantity) b; and, by ſuppoſing the denominator $= o$, x be likewiſe found equal to b; it may, in ſuch caſe, happen,

happen, that m ſhall be $= o$. When it ſo happens, the ordinate correſponding to the abſciſſa b will meet the curve in a double or triple &c. point, according as q has two or three &c. values. If, m being $= o$, q has two equal values, the ſaid ordinate will correſpond to a point of the curve where two branches thereof touch each other, either forming a cuſpid there, or being continued on both ſides of ſuch ordinate: and, if the values of q are all imaginary, ſuch ordinate will correſpond to a conjugate point.

The poſition of the tangents to the ſeveral branches of the curve, at ſuch double or triple &c. point, will be known by means of the ſeveral values of q: for, m being $= o$, q is the value of $[x \perp y]$ when x is equal to b; and therefore q will then be to unity, as the ordinate to the ſubtangent.

COROLLARY II. m being a poſitive odd number, or a poſitive fraction whoſe numerator and denominator are both odd numbers, as many ſingle different real values as q has, ſo many different branches of the curve are continued on both ſides of the ordinate d, touching each other in P, the point to which the ſaid ordinate correſponds: and two equal values of q denote two continued arches of equal curvature at P, and both turned one way; or a cuſpid of the ſecond kind, at that point.

The tangent, at P, to ſuch cuſpid, or continued arches, will be parallel to the baſe; and, according as q is negative or poſitive, the branch to which it relates will, at P, have its convexity upwards or downwards.

Imaginary values of q indicate a conjugate point at P; m being as in this, or the next Corollary.

COROLLARY III. m being a negative odd number, or a negative fraction whoſe numerator and denominator are both odd numbers, as many ſingle different real values as q has, ſo many different branches of the curve are continued on both ſides of the ordinate d, each forming a cuſpid of the firſt kind, at P: and two equal values of q denote two ſuch cuſpids, both pointing one way; or a cuſpid of the ſecond kind, at that point.

The tangent to each of theſe cuſpids will coincide with the ordinate; and, according as q is negative or poſitive, the cuſpid to which it relates will point upwards or downwards.

Corollary IV. m being an even number, or a fraction whoſe numerator is even and denominator odd, as many ſingle different real values as q has, ſo many different branches of the curve are continued on both ſides of the ordinate d, each having a point of contrary flexure at P: and two equal values of q denote two ſuch branches, both turned alike; or a cuſpid of the ſecond kind, at that point.

The tangent to ſuch cuſpid, or point of contrary flexure, will be parallel to the baſe, or coincident with the ordinate, according as m is poſitive or negative; and, according as q is poſitive or negative, y will be greater or leſs than d, when x is taken greater than b.

Corollary V. m being a poſitive fraction whoſe numerator is an odd number and denominator an even number, as many ſingle different poſitive real values as q and $\overline{-1}|^{m} \times q$ have, ſo many cuſpids of the firſt kind will be formed at P: and two equal poſitive real values of q, or of $\overline{-1}|^{m} \times q$, denote two ſuch cuſpids, at that point, both turned one way; or a cuſpid of the ſecond kind there.

The tangents to ſuch cuſpids will be parallel to the baſe; and, (x being ſuppoſed poſitive when the abſciſſa is on the right of the point where it begins,) according as the value of q or $\overline{-1}|^{m} \times q$ is real, the branch or branches forming the cuſpid, to which ſuch value relates, will be on the right or left of the ordinate d.

m being as in this, or the following Corollary, a conjugate point is indicated, if neither q nor $\overline{-1}|^{m} \times q$ be real.

Corollary VI. m being a negative fraction whoſe numerator is odd and denominator even, as many ſingle different poſitive real values as q and $\overline{-1}|^{m} \times q$ have, ſo many continued arches will be touched by the ordinate d: and two equal poſitive real values of q, or of $\overline{-1}|^{m} \times q$, denote, that d is a tangent to a cuſpid of the ſecond kind, at P; or that d touches two continued arches of equal curvature at that point, and both turned one way.

Moreover,

Moreover, (x being ſuppoſed poſitive when the abſciſſa is on the right of the point where it begins,) according as the value of q or $\overline{-1}|^{m} \times q$ is real, the continued arch, or the branches forming the cuſpid, to which ſuch value relates, will be on the right or left of the ſaid ordinate d.

SCHOLIUM. m being any number or fraction whatever, if q has two equal values, P, inſtead of being as above ſpecified, will ſometimes be a conjugate point. And, from this obſervation and what is ſaid above, the conſequence is obvious, when, m being of any value whatever, q has three or more equal values.

Theſe Corollaries are of great uſe, for preventing miſtakes concerning the greateſt and leaſt ordinates, and for aſcertaining the true form of a curve from the equation thereof; as will farther appear by the examples in the following articles.

5.

If the value of y be not expreſſed in terms of x, (as is frequently the caſe,) the value of $[x \perp y]$ will be expreſſed in terms containing both x and y. In which caſe, m and q will not be found as when $[x \perp y]$ is a function of x without y being concerned therein. But they may then be found as follows.

Suppoſing b to be a value of x, and d the correſpondent value of y, when either the numerator or denominator of the value of $[x \perp y]$ is $= o$; ſubſtitute $d + \overline{x - b}|^{m+1} \times Q$ for y, in the equation of the curve; (the terms being all on one ſide, and conſequently $= 0$;) and, in the expreſſion which reſults upon ſuch ſubſtitution, take m of ſuch a value, (greater than -1,) that the ſaid expreſſion being divided by the loweſt power of $x - b$ in the ſeveral members thereof, the quotient ſhall not, upon taking x equal to b, conſiſt of leſs than two members, all the terms in which Q is not concerned being accounted but one member. Then, ſuch quotient being $= o$, by writing therein q inſtead of Q, and the value of b inſtead of x, q will from thence be determined.

m and q may alſo be found by ſubſtituting $d + \overline{x - b}|^{m+1} \times q$ for y in the value of $[x \perp y]$. For the reſulting expreſſion being divided

divided by $\overline{m+1} \times \overline{x-b}|^m$, the quotient, when x is therein taken equal to b, will, by what is proved above, be equal to q: i. e. if R be put for the expreſſion which is obtained by ſubſtituting, as juſt now mentioned, in the value of $[x \perp y]$,

$$\frac{R}{\overline{x-b}|^m}, \text{ when } x \text{ is } = b, \text{ will be } = \overline{m+1} \times q:$$

from whence m and q may be readily determined, by taking m of ſuch a value, that the numerator and denominator of the value of the expreſſion $\frac{R}{\overline{x-b}|^m}$ ſhall each be diviſible by one and the ſame power of $x-b$; and, when each is divided by ſuch power, neither of the quotients ſhall vaniſh upon taking x equal to b.

REMARK. The exponents of the ſeveral powers of x and y being poſitive integers, the concluſion will be exactly the ſame, (and the proceſs more conciſe,) if, when the numerator of the value of $[x \perp y]$ vaniſhes, and the denominator, at the ſame time, does not vaniſh, we write b for x in the denominator, and only d for y in both numerator and denominator: Alſo, if, when the ſaid denominator vaniſhes, and the numerator, at the ſame time, does not vaniſh, we firſt write b for x in both numerator and denominator, and then ſubſtitute $d + \overline{x-b}|^{m+1} \times q$ for y in the denominator, and only d for y in the numerator; omitting (in this latter caſe) in the denominator, every term wherein y is not concerned before ſubſtitution, and every term but thoſe wherein the loweſt power of q is concerned after ſubſtitution.

For the terms omitted in the value of R in conſequence of theſe rules would, if they were therein written, always vaniſh in carrying on the proceſs; as will eaſily appear, upon conſidering what we juſt now ſaid concerning the expreſſion $\frac{R}{\overline{x-b}|^m}$.

EXAMPLE I. *Let the equation of the curve be* $x^4 - axy^2 + cy^3 = 0$, a *and* c *being invariable poſitive quantities.*

Then, by reſidual diviſion, we have $4x^3 - ay^2 - 2axy[x \perp y] + 3cy^2[x \perp y] = 0$; and conſequently

$$[x \perp y] = \frac{4x^3 - ay^2}{2axy - 3cy^2}.$$

Suppoſing

Supposing the numerator $4x^3 - ay^2 = o$, we, from thence and the equation of the curve, find $x = o$, also $x = \frac{3^2a^3}{4^3c^2}$; and the respective values of y are o, and $\frac{3^3a^4}{4^4c^3}$.

Supposing the denominator $2axy - 3cy^2 = o$, we, from hence and the given equation, find $x = o$, also $x = \frac{2^2a^3}{3^3c^2}$; and the respective values of y are o, and $\frac{2^3a^4}{3^4c^3}$.

1st. Taking $b = o$, d will be $= o$, and $d + \overline{x - b}|^{m+1} \times Q$ (the quantity to be substituted for y, in the equation of the curve,) $= Qx^{m+1}$. Therefore $x^4 - ax^{2m+3}Q^2 + cx^{3m+3}Q^3$ will be $= o$; where, it is obvious, that, to determine q as taught above, m may be $= o$, also $m = \frac{1}{2}$.

If m be $= o$, $x^4 - ax^3Q^2 + cx^3Q^3$ will be $= o$. Therefore, in this case, (dividing by x^3, and then taking $x = o$,) $- aq^2 + cq^3$ is $= o$, or $q^3 - \frac{a}{c}q^2 = o$. Here q has three values, two of which are each $= o$, and the third $= \frac{a}{c}$.

Therefore, by Cor. 1. of the last Art. the point (A) where x begins is a triple point in the curve; and two of its branches touch each other at that point. Moreover, q being to unity, as the ordinate to the subtangent; and the value of q which relates to the said two branches being $= o$; the tangent to these branches at A coincides with the base AF: and, the value of q which relates to the third branch being $\frac{a}{c}$, the tangent (AG) to this branch at A makes an angle GAH with the base, such, that GH (which is supposed parallel to the ordinates of the curve) is to AH, as $\frac{a}{c}$ to 1. Fig. 51.

If m be equal to $\frac{1}{2}$, $x^4 - ax^4Q^2 + cx^{\frac{9}{2}}Q^3$ will be $= o$.

Hence

Hence $1 - aq^2 = 0$, and $q = \frac{1}{\sqrt{a}}$. Therefore, by Cor. 5. of the laſt Article, there is a cuſpid of the firſt kind at A, the tangent to which is coincident with the baſe; and (x being always ſuppoſed poſitive when the abſciſſa is on the right of the point at which it begins,) the branches forming ſuch cuſpid are on the right of the ſame point: which agrees with what we diſcovered by conſidering m equal to o.

2dly. Taking b equal to $\frac{3^2a^3}{4^3c^2}$, d will be $= \frac{3^3a^4}{4^4c^3}$. Therefore $\frac{R}{\overline{x-b}|^m}$ (abridged agreeable to our Remark) is $= \frac{4}{\overline{x-b}|^m} \times \frac{x^3 - \frac{3^6a^9}{4^9c^6}}{2abd - 3cd^2} = \frac{4}{\overline{x-b}|^m} \times \frac{x^3 - b^3}{2abd - 3cd^2} = \frac{4}{\overline{x-b}|^m} \times \frac{\overline{x-b} \times \overline{x^2 + bx + b^2}}{2abd - 3cd^2}$ $= \overline{m+1} \times q$ when x is $= b$. Whence it is manifeſt, that m is $= 1$, and $q = \frac{6b^2}{2abd - 3cd^2}$. Therefore, q being a negative quantity, it follows from what is ſaid above, that the ordinate BP ($= d$) is a *maximum* when (AB) the correſponding abſciſſa is equal to b.

3dly. Taking b equal to $\frac{2^2a^3}{3^3c^2}$, d will be $= \frac{2^3a^4}{3^4c^3}$. Therefore $\frac{R}{\overline{x-b}|^m}$ (abridged according to our Remark) is $= \overline{x-b}|^{-m} \times \frac{4b^3 - ad^2}{\overline{2ab - 6cd} \cdot q \cdot \overline{x-b}|^{m+1}} = \overline{m+1} \times q$ when x is $= b$. It appears therefore, that m is $= -\frac{1}{2}$, and $\frac{4b^3 - ad^2}{\overline{2ab - 6cd} \times q} = \frac{1}{2}q$: Hence q is found $= \sqrt{\frac{4b^3 - ad^2}{ab - 3cd}}$.

Now, $4b^3$ being greater than ad^2, and ab leſs than $3cd$, the value of q is imaginary, and the value of $\overline{-1}|^m \times q$ is real. Therefore, by Cor. 6. of the laſt Article, the ordinate ef ($= d$) touches a continued arch when (Ae) the correſponding abſciſſa is equal to b; and the arch ſo touched is on the left of the ſaid ordinate.

EXAMPLE

EXAMPLE II. *Let the equation of the curve be* $x^4 - 2x^2y - 4xy^2 + y^2 - y^3 = 0$.

Then, by residual division, we have $4x^3 - 4xy - 4y^2 - 2x^2[x \perp y] - 8xy[x \perp y] + 2y[x \perp y] - 3y^2[x \perp y] = 0$; and consequently $[x \perp y] = \frac{4x^3 - 4xy - 4y^2}{2x^2 + 8xy - 2y + 3y^2}$.

Supposing the numerator $4x^3 - 4xy - 4y^2 = 0$, we, from thence and the equation of the curve, find $x = 0$, also $x = 2$; and the respective values of y are 0, and -4; y being, in such case, $= \frac{x^2 - 2x^3}{1 - x + x^2}$.

Supposing the denominator $2x^2 + 8xy - 2y + 3y^2 = 0$, we, from hence and the given equation, find $x = 0$, or $x = 2$, or $x = \frac{4}{27}$; and the respective values of y are 0, -4, and $\frac{16}{81}$; y being, in this case, $= \frac{9x^4 + 8x^3 - 2x^2}{16x - 20x^2 - 2}$.

1st. Taking b equal to o, d will be $= o$, and $d + \overline{x - b}|^{m+1} \times Q = Qx^{m+1}$. Therefore, this last quantity being substituted instead of y in the equation of the curve, we have

$$x^4 - 2Qx^{m+3} - 4Q^2x^{2m+3} + Q^2x^{2m+2} - Q^3x^{3m+3} = 0:$$

And here, to determine q, m must be $= 1$.

Consequently $1 - 2Q - 4Q^2x + Q^2 - Q^3x^2$ is $= 0$; from whence we have $1 - 2q + q^2 = 0$: And q having two equal values, each $= 1$, there is, agreeable to Corol. 2. of the last Article, a cuspid of the second kind, at the point A, * where Fig. 52. x be-

* By Corollary 2. of the last Article, there is either a cuspid of the second kind, at the point A; or the base touches two continued arches, at that point. To know whether the curve has a cuspid there, or not, some farther enquiry concerning the value of Q is requisite.—Having found, that Q is $= 1$ when x is $= 0$, we may suppose $Q = 1 + x^n\overset{\prime\prime}{Q}$, n being some positive number or fraction, and $\overset{\prime\prime}{Q}$ such a function of x, that neither it nor its reciprocal shall vanish upon taking x equal to 0.

Let therefore $1 + x^n\overset{\prime\prime}{Q}$ be substituted instead of Q in the equation $1 - 2Q + Q^2 - 4Q^2x - Q^3x^2 = 0$: And then, from the resulting equation, n may be

x begins; the tangent to which coincides with the baſe, and the convexity of the branches forming ſuch cuſpid is downwards.

2dly. Taking b equal to 2, d will be $= -4$, and $x^4 + 8x^2 - 64x + 80 - 2 \times \overline{x^2 - 16x + 28} \times \overline{x-2}|^{m+1} \times Q + \overline{13 - 4x} \times \overline{x-2}|^{2m+2} \times Q^2 - \overline{x-2}|^{3m+3} \times Q^3 = 0$. It appears therefore, that $\overline{x^2 + 4x + 20} \times \overline{x-2}|^2 + \overline{28 - 2x} \times \overline{x-2}|^{m+2} \times Q + \overline{13 - 4x} \times \overline{x-2}|^{2m+2} \times Q^2 - \overline{x-2}|^{3m+3} \times Q^3$ is $= 0$; and that, to determine q, m muſt be $= 0$. Conſequently (dividing by $\overline{x-2}|^2$, and afterwards taking $x = 2$, and $Q = q$,) $32 + 24q + 5q^2$ is $= 0$, and $q = -\frac{12}{5} \pm \frac{4}{5}\sqrt{-1}$. Here, theſe values of q being imaginary, the ordinate d, by Cor. 1. of the laſt Art. correſponds to a conjugate point, when the abſciſſa is $= 2$.

3dly. Taking b equal to $\frac{4}{27}$, d will be $= \frac{16}{81}$. Therefore $\frac{R}{\overline{x-b}|^m}$ (abridged agreeable to our Remark) is $= \overline{x-b}|^{-m} \times \frac{4b^3 - 4bd - 4d^2}{\frac{10}{27}q \cdot \overline{x-b}|^{m+1}} = \overline{m+1} \times q$ when x is $= b$. It is evident therefore, that m is $= -\frac{1}{2}$, and $\frac{4b^3 - 4bd - 4d^2}{\frac{10}{27}q} = \frac{1}{2}q$: Hence q is found $= \sqrt{\frac{108}{5} \times \overline{b^3 - bd - d^2}}$.

---

be determined, and alſo $\overset{\prime\prime}{q}$, the value of $\overset{\prime\prime}{Q}$ when x is $= 0$, in the ſame manner as m and q are determined above.

By proceeding in that manner, n is found $= \frac{1}{2}$, and $\overset{\prime\prime}{q} = \pm 2$. Conſequently y $(= Qx^2)$ is $= x^2 + x^{\frac{5}{2}}\overset{\prime\prime}{Q}$, $\overset{\prime\prime}{Q}$ being ſuch a function of x, that its value is ± 2 when x is $= 0$. Whence ($x^{\frac{5}{2}}\overset{\prime\prime}{Q}$ being imaginary when x is negative,) it appears, that the curve is not continued on the left of the point A. Therefore, by what is ſaid above, there muſt be a cuſpid of the ſecond kind, at A, as in Fig. 52.

And by purſuing the ſame method, we may be ſatisfied in other ſuch ambiguous caſes.

Conſe-

Consequently, the value of q being imaginary, the value of $\overline{-1}|^m \times q$ is real: and, by Corol. 6. of the last Art. the ordinate BD ($= d$) touches a continued arch, when (AB) the correspondent abscissa is equal to b; and the arch so touched is on the left of the said ordinate.

EXAMPLE III. *Suppose* $x^3 - 3xy^2 + y^5 = 0$.

Then will $[x \dotdiv y]$ be $= \frac{3x^2 - 3y^2}{6xy - 5y^4}$.

Supposing the numerator $3x^2 - 3y^2 = 0$, we, from thence and the other equation, find $x = 0$, or $x = \sqrt{2}$, or $x = -\sqrt{2}$; and the respective values of y are 0, $\sqrt{2}$, and $-\sqrt{2}$.

Supposing the denominator $6xy - 5y^4 = 0$, we, from hence and the first equation, find $x = 0$, or $x = \frac{9}{5^{\frac{5}{4}}} \times \sqrt{2}$, or $x = -\frac{9}{5^{\frac{5}{4}}} \times \sqrt{2}$, and the respective values of y are 0, $\frac{3}{5^{\frac{3}{4}}} \times \sqrt{2}$, and $-\frac{3}{5^{\frac{3}{4}}} \times \sqrt{2}$.

1st. Taking b equal to 0, d will be $= 0$, and $d + \overline{x - b}|^{m+1} \times Q = Qx^{m+1}$. Therefore $x^3 - 3Q^2x^{2m+3} + Q^5x^{5m+5}$ is $= 0$; where, it is obvious, m may be $= 0$, also $m = -\frac{2}{3}$.

If m be equal to 0, $1 - 3q^2$ will be $= 0$: Hence q is found $= \overline{\frac{1}{3}}|^{\frac{1}{2}}$, or $q = -\overline{\frac{1}{3}}|^{\frac{1}{2}}$. By these two values of q it appears, that two branches of the curve intersect each other at A, the point where x begins: and that the tangents (AG, Ag) to those branches, at A, make equal angles (GAH, gAh) with the base AE; and such that GH and gh (which are supposed parallel to the ordinates of the curve) are to AH and Ah respectively, as $\overline{\frac{1}{3}}|^{\frac{1}{2}}$ to 1. Fig. 53.

If m be equal to $-\frac{2}{3}$, $-3q^2 + q^5$ will be $= 0$: Hence we have $q = 3^{\frac{1}{3}}$. Therefore, by Corollary 4. of the last Article,

A is a point of contrary flexure, in another branch of the curve; the tangent to which, at that point, is parallel to the ordinates; and, q being positive, the value of y correspónding to this branch will be positive when x is positive.

2dly. Taking b equal to $\pm\sqrt{2}$, d will be $= +\sqrt{2}$. Therefore $\frac{R}{\overline{x-b}|^m}$ (abridged according to our Remark) is $=$

$$\frac{3}{\overline{x-b}|^m} \times \frac{x^2-2}{6bd-5d^4} = \frac{3}{\overline{x\mp\sqrt{2}}|^m} \times \frac{\overline{x-\sqrt{2}} \times \overline{x+\sqrt{2}}}{6bd-5d^4} = \overline{m+1} \times q$$

when x is $= b$. Whence it is evident, that m is $= 1$, and $q = \frac{3b}{6bd-5d^4}$. Therefore, q being negative when b is $= \sqrt{2}$ and positive when b is $= -\sqrt{2}$, it follows, from what is said above, that the ordinate BP ($= \sqrt{2}$) is a *maximum* when (AB) the correspondent abscissa is $= \sqrt{2}$: and that the ordinate bp ($= \sqrt{2}$) is a *minimum* when (Ab) the abscissa corresponding thereto is equal to $\sqrt{2}$; b being on the contrary side of A, from B; and the point p below the base.

3dly. Taking b equal to $\pm \frac{9}{5^{\frac{5}{4}}} \times \sqrt{2}$, d will be $= \pm \frac{3}{5^{\frac{3}{4}}} \times \sqrt{2}$. Therefore $\frac{R}{\overline{x-b}|^m}$ (abridged agreeable to our Remark) is $=$

$$\overline{x-b}|^{-m} \times \frac{3b^2-3d^2}{-15d^3q \,.\, \overline{x-b}|^{m+1}} = \overline{m+1} \times q \text{ when } x \text{ is } = b.$$

Whence it is manifest, that m is $= -\frac{1}{2}$, and $q = \overline{\frac{2b^2-2d^2}{-5d^3}}\Big|^{\frac{1}{2}}$. Consequently, the value of q being imaginary or real, and the value of $\overline{-1}|^m \times q$ real or imaginary, according as b is taken equal to $\frac{9}{5^{\frac{5}{4}}} \times \sqrt{2}$ or $-\frac{9}{5^{\frac{5}{4}}} \times \sqrt{2}$; the ordinate EF (by Cor. 6. of the last Art.) touches a continued arch, when (AE) the abscissa answering thereto is equal to $\frac{9}{5^{\frac{5}{4}}} \times \sqrt{2}$, and the arch so touched is on the left of the said ordinate: likewise the ordinate ef touches a continued arch, when (Ae) the correspondent abscissa is

is equal to AE; e being on the contrary ſide of A, from E; and the point f below the baſe; and the arch ſo touched is on the right of the ſaid ordinate ef.

6.

When m is $= o$, and the curve is continued on both ſides of the ordinate d, that ordinate may paſs through a point of contrary flexure, the tangent to which is oblique to the baſe and ordinate.—Suppoſe P to be ſuch a point in the curve Pf, and let Fig. 54.
Pg be a tangent at that point. Then, the ordinate ef, interſect- 55.
ing that tangent in g, being called y; the ordinate BP, d; and the abſciſſas AB and Ae, b and x reſpectively; $ef - eg$ will be $= y - d - \overline{x - b} \times q$, q denoting the value of $[x \perp y]$ when x is $= b$, i. e. q is $= d \div$ Subtang. Which value of $ef - eg$, it is manifeſt, will (as when it relates to Fig. 54.) be poſitive when x is greater than b, and negative when x is leſs than b; or (as when it relates to Fig. 55.) poſitive when x is leſs than b, and negative when x is greater than b. Therefore, by Chap. 4. Art. 3. $\overline{x - b}|^n \overset{\prime\prime}{Q}$ may be aſſumed $= y - d - \overline{x - b} \times q$; n being an odd number, or a fraction whoſe numerator and denominator are both odd numbers; and $\overset{\prime\prime}{Q}$ ſuch a function of x, that neither it nor its reciprocal ſhall vaniſh upon taking x equal to b. From which aſſumed equation we have $y = d + \overline{x - b} . q + \overline{x - b}|^n . \overset{\prime\prime}{Q}$. Therefore, y being before ſuppoſed $= d + \overline{x - b} . Q$, it is evident $q + \overline{x - b}|^{n-1} \times \overset{\prime\prime}{Q}$ will be $= Q$; where, it is plain, n muſt be greater than 1.

Let therefore $q + \overline{x - b}|^{n-1} \times \overset{\prime\prime}{Q}$ be ſubſtituted inſtead of Q, in any equation found by ſubſtituting, and taking m equal to o, as in the preceding article: And then, ſuppoſing $\overset{\prime\prime}{q}$ to be the value of $\overset{\prime\prime}{Q}$ when x is $= b$, n and $\overset{\prime\prime}{q}$ will be determined in the ſame manner as are m and q in that article.

By the value of n (ſo determined) it will appear whether the ordinate d correſponds to a point of contrary flexure, or not; and,

and, if it does correspond to such a point, the value of $\ddot{q}$ will shew the tendency of the curve on both sides thereof, the convexity of the branch on the right of that ordinate being downwards or upwards, according as $\overset{\prime\prime}{q}$ is positive or negative.

EXAMPLE. *In Examp.* 3. *of the last article,* y *is* $= 0$, *when* x *is* $= 0$: and by substituting Qx^{m+1} instead of y, in the equation of the curve, we get $x^3 - 3Q^2x^{2m+3} + Q^5x^{5m+5} = 0$; from whence, by taking m equal to 0, and dividing by x^3, we have $1 - 3Q^2 + Q^5x^2 = 0$. Hence q (the value of Q when x is $= 0$) is found $= + \sqrt{\frac{1}{3}}$.

Substituting $q + x^{n-1}\overset{\prime\prime}{Q}$ instead of Q, we have

$$\left.\begin{array}{l} 1 - 3q^2 - 6qx^{n-1}\overset{\prime\prime}{Q} - 3x^{2n-2}\overset{\prime\prime}{Q}^2 \\ + q^5x^2 + 5q^4x^{n+1}\overset{\prime\prime}{Q} + 10q^3x^{2n}\overset{\prime\prime}{Q}^2 + 10q^2x^{3n-1}\overset{\prime\prime}{Q}^3 + \\ \quad 5qx^{4n-2}\overset{\prime\prime}{Q}^4 + x^{5n-3}\overset{\prime\prime}{Q}^5 \end{array}\right\} = 0.$$

Here, it is obvious, n must be $= 3$; and consequently $q^5 - 6q\overset{\prime\prime}{q} = 0$.

Therefore $\overset{\prime\prime}{q}$ is $= \frac{q^4}{6}$; $(= \frac{1}{54}$, as well when q is $= - \sqrt{\frac{1}{3}}$, as when q is $= + \sqrt{\frac{1}{3}}$;) and it appears that the two branches of

Fig. 53. the curve, whose tangents at the point A are AG and Ag, have each a point of contrary flexure at A, as in Fig. 53.

7.

Fig. 54. 55. P being a point of contrary flexure, and the abscissa and ordinate corresponding thereto being called β and δ respectively; and the value of $[x \lrcorner y]$, when x is $= \beta$, being denoted by q; $y - \delta - \overline{x - \beta} . q$, as is shewn in the preceding article, will be positive when x is greater than β, and negative when x is less than β; or positive when x is less than β, and negative when x is

is greater than β, how near ſoever x be taken to β. Therefore, $\frac{y - \delta - \overline{x - \beta}\,.\,q}{\overline{x-\beta}|^2}$ being poſitive or negative according as $y - \delta - \overline{x - \beta}\,.\,q$ is poſitive or negative, it follows from Art. 1. Chap. 4. that the value of $\frac{y - \delta - \overline{x - \beta}\,.\,q}{\overline{x-\beta}|^2}$, or its reciprocal, will be $= o$, when x is $= \beta$.

Now, by what is taught in Chap. 2. the value of $\frac{y - \delta - \overline{x - \beta}\,.\,q}{\overline{x-\beta}|^2}$, when x is $= \beta$, (i. e. when y is $= \delta$,) is $=$ the value of $\frac{[x \ddot{-} y]}{2}$.

Conſequently $[x \ddot{-} y]$ is $= o$, or $\frac{1}{[x \ddot{-} y]} = o$, when the ordinate y correſponds to a point of contrary flexure: And the ſame concluſion follows, from conſidering that $[x \ddot{-} y]$, the quotient of the ordinate divided by the ſubtangent, is then a *minimum*, as in Fig. 54. or a *maximum*, as in Fig. 55.

Moreover, ſuppoſing β to be a value of x, and δ the correſpondent value of y, in either of the equations $[x \ddot{-} y] = o$, or $\frac{1}{[x \ddot{-} y]} = o$; and aſſuming $\overline{x - \beta}|^n \times \ddot{Q} = \frac{y - \delta - \overline{x - \beta}\,.\,q}{\overline{x-\beta}|^2}$, $\ddot{Q}$ being ſuch a function of x, that neither $\ddot{q}$, the value of $\ddot{Q}$ when x is $= \beta$, nor the reciprocal of that value ſhall vaniſh; we have $\frac{y - \delta - \overline{x - \beta} \times q}{\overline{x-\beta}|^{n+2}} = \ddot{Q}$; from whence, when x is $= \beta$, $\frac{[x \dot{-} y] - q}{n + 2\,.\,\overline{x - \beta}|^{n+1}} = \frac{[x \ddot{-} y]}{n + 1\,.\,n + 2\,.\,\overline{x-\beta}|^n} = \ddot{q}$. And it follows, from what is before ſaid, in this and the 4th Chap. that, the curve being continued on both ſides of the ordinate δ, if n be an odd number, or a fraction whoſe numerator and denominator are both odd numbers, $[x \dot{-} y]$ (when x is $= \beta$) will be a *minimum* or a *maximum*, according as $\ddot{q}$ is poſitive or negative; and the ſaid ordinate δ ſhall paſs through a point of contrary flexure. But if n be an even number, or a fraction whoſe numerator is even and denominator odd, ſuch ordinate will correſpond to a point of the curve where the curvature is *nothing*

nothing or *infinite*, and the concavity of the curve is turned the ſame way on both ſides of that point.

When $[x \stackrel{..}{-} y]$ is a function of x without y being concerned therein, n and $\overset{\prime\prime}{q}$ will be determined in like manner as are m and q when $[x \stackrel{.}{-} y]$ is ſuch a function of x.

To find n and $\overset{\prime\prime}{q}$ when the value of $[x \stackrel{..}{-} y]$ conſiſts of terms in which both x and y are concerned, let $\delta + \overline{x - \beta} \,.\, q + \overline{x - \beta}|^{n+2} \times \overset{\prime\prime}{Q}$ be ſubſtituted for y, in the equation of the curve; or $\delta + \overline{x - \beta} \,.\, q + \overline{x - \beta}|^{n+2} \times \overset{\prime\prime}{q}$ for y in either of the equations $\dfrac{[x \stackrel{.}{-} y] - q}{\overline{n+2} \,.\, \overline{x - \beta}|^{n+1}} = \overset{\prime\prime}{q}$ when x is $= \beta$, or $\dfrac{[x \stackrel{..}{-} y]}{\overline{n+1} \,.\, \overline{n+2} \,.\, \overline{x - \beta}|^{n}} = \overset{\prime\prime}{q}$ when x is $= \beta$. From either of which equations, after ſubſtitution, n and $\overset{\prime\prime}{q}$ may be determined in the ſame manner as m and q are determined above.

Corollary I. n being an odd number, or a fraction whoſe numerator and denominator are both odd numbers, as many ſingle different real values as $\overset{\prime\prime}{q}$ has, ſo many different branches of the curve are continued on both ſides of the ordinate δ, each having a point of contrary flexure at P, the point to which the ſaid ordinate correſponds: And two equal values of $\overset{\prime\prime}{q}$ denote two ſuch branches, both turned alike; or a cuſpid of the ſecond kind, at that point.

Moreover, (x being ſuppoſed poſitive when the abſciſſa is on the right of the point where it begins) the branch of the curve on the right of the ordinate δ will have its convexity downwards or upwards, according as $\overset{\prime\prime}{q}$ is poſitive or negative.

Corollary

COROLLARY II. n being an even number, or a fraction whose numerator is even and denominator odd, as many single different real values as $\overset{''}{q}$ has, so many different branches of the curve are continued on both sides of the ordinate δ, touching each other in P, and each having its curvature *nothing* or *infinite*, at that point, according as n is positive or negative: And two equal values of $\overset{''}{q}$ denote two such continued arches, both turned one way; or a cuspid of the second kind, at P.

Moreover, according as $\overset{''}{q}$ is positive or negative, the branch to which it relates will, at P, have its convexity downwards or upwards.

COROLLARY III. n being a fraction whose numerator is odd and denominator even, as many single different positive real values as $\overset{''}{q}$ and $\overline{-1}|^n \times \overset{''}{q}$ have, so many cuspids of the first kind will be formed at P: and two equal positive real values of $\overset{''}{q}$, or of $\overline{-1}|^n \times \overset{''}{q}$, denote two such cuspids, at that point, both pointing one way; or a cuspid of the second kind there.

Moreover, (x being supposed positive when the abscissa is on the right of the point where it begins,) according as the value of $\overset{''}{q}$ or $\overline{-1}|^n \times \overset{''}{q}$ is real, the branch or branches forming the cuspid to which such value relates will be on the right or left of the ordinate δ.

COROLLARY IV. n being $= 0$, as many single different real values as $\overset{''}{q}$ has, so many different branches of the curve are continued on both sides of the ordinate δ, touching each other in P: and two equal values of $\overset{''}{q}$ denote two continued arches of equal curvature at P, and both turned one way; or a cuspid of the second kind, at that point.

The tangent, at P, to ſuch cuſpid, or continued arches, will be oblique to the baſe and ordinate; and, according as $\ddot{q}$ is poſitive or negative, the branch to which it relates will, at P, have its convexity downwards or upwards.

Having ſaid ſo much concerning curves referred to a baſe, *I think it unneceſſary to add any thing here relating to* SPIRALS;—*to which kind of curves the intelligent Reader will readily apply the above method of reaſoning.*

THE

RESIDUAL ANALYSIS.

CHAP. IX.

Of the ASYMPTOTES *of* curve Lines.

AS a knowledge of the tendency of the infinite branches of a Curve, when ſuch there are, is requiſite for the obtaining a clear idea of the figure of ſuch Curve, and may ſometimes facilitate the buſineſs of finding the moſt diſtinguiſhed points therein; I ſhall therefore point out an eaſy method of finding the Aſymptotes to ſuch branches, by which means their tendency may be very readily diſcovered.

A branch of a curve when infinitely continued may be conſidered as coinciding with its aſymptote. If therefore m and n be reſpectively put for the ſines of the angles which a rectilinear aſymptote makes with any ordinate and the baſe; or for the ſines of the angles which the tangent to a curvilinear aſymptote, when extended infinitely, makes with any ordinate and baſe; m will be to n, as $[v \dot{-} x]$ to $[v \dot{-} y]$, when the curve is infinitely continued. Conſequently, if from the equation of the propoſed curve, (which ſuppoſe clear of ſurds,) we, by reſidual diviſion, ($v - v_{,}$ being the diviſor,) deduce a ſecond equation, and therein write m and n inſtead of $[v \dot{-} x]$ and $[v \dot{-} y]$ reſpectively, we

ſhall obtain the equation of an aſymptotic line, one dimenſion lower than the given equation, where m and n will be ſome invariable quantities, which will be determined in conſequence of this proceſs. And, the propoſed curve being above the ſecond order of lines, we, from the equation of that firſt aſymptotic line, by the like reſidual diviſion, and again ſubſtituting m and n inſtead of $[v \mathbin{\raisebox{0.3ex}{$\cdot$}\!\!-} x]$ and $[v \mathbin{\raisebox{0.3ex}{$\cdot$}\!\!-} y]$ reſpectively, may deduce the equation of a ſecond aſymptotic line, two dimenſions lower than the given equation. Likewiſe, the propoſed curve being above the third order of lines, we, by our ſaid diviſion and ſubſtituting as before, may derive, from the equation of ſuch ſecond aſymptotic line, the equation of a third aſymptotic line, three dimenſions lower than the given equation. And ſo, with great facility, we may proceed with any given equation, till we get equations of aſymptotic lines of every dimenſion lower than the given equation: alſo, from the loweſt of thoſe equations, (viz. that of one dimenſion,) we may, by continuing the ſame proceſs, obtain an equation containing no other unknown quantities but m and n; from whence $\frac{m}{n}$, or $\frac{n}{m}$ will be determined; and then all that is requiſite, in the ſeveral equations of the aſymptotic lines before found, will be known; and conſequently the tendency of ſuch lines, and of ſuch branches of the propoſed curve to which they are aſymptotes.

Scholium I. The poſition of the ultimate tangent * to a parabola being not determinable, ſuch tangent only coming, at an unlimited diſtance, almoſt to a paralleliſm with ſome certain right line: when the aſymptotic line (expreſſed by any equation found as above) is of the parabolic kind, it will appear by ſome equation in the above proceſs, that m (or n) cannot be taken abſolutely equal to its apparent value in the final equation in the ſaid proceſs, without occaſioning an abſurdity; and the impoſſibility of a rectilinear aſymptote will be evident. Therefore, in ſuch caſe, we are to conſider m (or n) only as approaching very near to ſuch apparent value; and accordingly, inſtead of $\frac{m}{n}$ (or $\frac{n}{m}$) ſubſtitute its value obtained from ſome equation before found.

* By *ultimate tangent*, I mean the tangent to an infinitely diſtant point of the curve.

Scho-

SCHOLIUM II. It may happen that the aſymptotic equation of two (or more) dimenſions, found by our proceſs, will expreſs two (or more) right lines, inſtead of expreſſing a curve. In which caſe, to find the moſt ſimple hyperbolic aſymptote of the branches to which a rectilinear one ſo found relates, ſome farther enquiry is requiſite. Suppoſe the equation of a rectilinear aſymptote (found by the above proceſs) to be $y = Ax + B$. Then, it is obvious, the branches to which ſuch right line is an aſymptote may be expreſſed by the equation $y = Ax + B + x^p Q$; p being ſome negative number or fraction, and Q ſuch a function of x, that neither (q) the limit to which it converges, when x (being a very large quantity) is taken greater and greater, nor $\left(\frac{1}{q}\right)$ the reciprocal of that limit, ſhall be $= 0$. And it is likewiſe obvious, that $y = Ax + B + qx^p$ is the equation of the moſt ſimple curve which may be an aſymptote to ſuch branches. Therefore let $Ax + B + x^p Q$ be ſubſtituted for y in the given equation; and take p of ſuch a negative value in the reſulting equation; that the ſame, after it is divided by the higheſt power of x therein concerned, ſhall not, upon ſuppoſing x infinitely great, conſiſt of leſs than two finite members: From whence q may be determined.

EXAMPLE I. *Let the equation of the curve be* $y^4 - ax^2y^2 + bx^3 = 0$; a *and* b *being invariable poſitive quantities.*

Then, by proceeding as above mentioned, we find the

$$\text{1ſt } \textit{aſympt. equat.}\ 4ny^3 \begin{array}{l} -\,2amxy^2 \\ -\,2anx^2y \end{array} + 3bmx^2 = 0;$$

$$\text{2d } \textit{aſympt. equat.}\ 6n^2y^2 \begin{array}{l} -\,am^2y^2 \\ -\,4amnxy \\ -\,an^2x^2 \end{array} + 3bm^2x = 0;$$

$$\text{3d } \textit{aſympt. equat.}\ 4n^3y \begin{array}{l} -\,2am^2ny \\ -\,2amn^2x \end{array} + bm^3 = 0:$$

$$\text{alſo } n^4 - am^2n^2 = 0;$$

$$\text{where } \frac{n}{m} \text{ is } = 0, \text{ or } \frac{n}{m} = \pm\sqrt{a}.$$

Taking

Taking $\frac{n}{m}$ equal to $+a^{\frac{1}{2}}$ and $-a^{\frac{1}{2}}$ successively, it appears, by the 3d asympt. equat. that the curve has two rectilinear asymptotes, whose equations are $2a^{\frac{3}{2}}y - 2a^2x + b = 0$, and $-2a^{\frac{3}{2}}y - 2a^2x + b = 0$; and, by the 2d asympt. equat. that the two hyperbolas, whose equations are $5ay^2 - 4a^{\frac{3}{2}}xy - a^2x^2 + 3b = 0$, and $5ay^2 + 4a^{\frac{3}{2}}xy - a^2x^2 + 3b = 0$, are asymptotes of the proposed curve; also, by the 1st asympt. equat. that the curve has for asymptotes two lines of the third order, whose equations are $4a^{\frac{1}{2}}y^3 - 2axy^2 - 2a^{\frac{3}{2}}x^2y + 3bx^2 = 0$, and $-4a^{\frac{1}{2}}y^3 - 2axy^2 + 2a^{\frac{3}{2}}x^2y + 3bx^2 = 0$.

Taking $\frac{n}{m}$ equal to 0, we have, from the 3d asympt. equat. $b = 0$: which is absurd, (as b is supposed of some value,) and therefore, according to Schol. 1. shews that there is no rectilinear asymptote parallel to the base; (the final equation $\frac{n}{m} = 0$ only indicating, that the ultimate tangent of some parabolic asymptote has nearly that direction;) and that, for the said asymptotic equation to be a true one, $\frac{n}{m}$ must not be considered as absolutely $= 0$, but only as indefinitely small, so that being multiplied by a very great quantity, the product may be something considerable (viz. $= b$). Therefore, from that consideration, retaining the terms wherein the root $\frac{n}{m}$ is found, (after dividing by m^3,) as being most considerable; and rejecting those in which the higher powers of $\frac{n}{m}$ are concerned, as inconsiderable; we have $\frac{2any}{m} = b$, or $\frac{n}{m} = \frac{b}{2ay}$. Which last quantity being substituted for $\frac{n}{m}$ in the second asymptotic equation, after dividing by m^2 and rejecting the terms wherein $\frac{n^2}{m^2}$ is concerned, we get $bx - ay^2 = 0$. Consequently the conical parabola expressed by this last equation, is an asymptote to the proposed curve. Moreover, by substituting $\frac{b}{2ay}$ instead of $\frac{n}{m}$ in the 1st asympt. equat. it appears that a

line

line of the third order, whose equation is $by^3 - a^2xy^2 + abx^2 = 0$, is an asymptote to the same curve.

EXAMPLE II. *Let the equation of the curve be*

$$xy^2 + ey - ax^3 - bx^2 - cx - d = 0:$$

Which is the chief equation in Sir ISAAC NEWTON'S *Enumerat. Linear. Tert. Ordinis.*

Then, by residual division, we find the

1st *asympt. equat.* $\begin{matrix} my^2 \\ +2nxy \end{matrix} + en - 3amx^2 - 2bmx - cm = 0;$

2d *asympt. equat.* $\begin{matrix} 2mny \\ +n^2x \end{matrix} - 3am^2x - bm^2 = 0:$

$$\text{also } mn^2 - am^3 = 0;$$

$$\text{where } \frac{m}{n} \text{ is } = 0, \text{ and } \frac{m}{n} = \pm\frac{1}{a^{\frac{1}{2}}}.$$

It is evident therefore, that, when a is a positive quantity, the curve has three rectilinear asymptotes; one of which, as appears by the second asymptotic equation, is the principal ordinate, and the other two are expressed by the equations $2a^{\frac{1}{2}}y - 2ax - b = 0$, and $2a^{\frac{1}{2}}y + 2ax + b = 0$. Moreover, by taking $\frac{m}{n}$ equal to 0, $\frac{1}{a^{\frac{1}{2}}}$, and $-\frac{1}{a^{\frac{1}{2}}}$ successively, in the 1st asympt. equat. it appears that the curve has for asymptotes the three hyperbolas, whose equations are $2xy + e = 0$, $y^2 + 2a^{\frac{1}{2}}xy + a^{\frac{1}{2}}e - 3ax^2 - 2bx - c = 0$, and $y^2 - 2a^{\frac{1}{2}}xy - a^{\frac{1}{2}}e - 3ax^2 - 2bx - c = 0$.

When a is $= 0$, mn^2 is $= 0$; whence $\frac{m}{n} = 0$, or $\frac{n}{m} = 0$. Therefore, if b be not also $= 0$, it is manifest, from the second asympt. equat. that the curve cannot have a rectilinear asymptote parallel to the base: But (rejecting $\frac{n^2}{m^2}x$ as inconsiderable) we

have,

have, from that aſymptotic equation, $\frac{n}{m} = \frac{b}{2y}$. Which laſt quantity being ſubſtituted for $\frac{n}{m}$, in the 1ſt aſympt. equat. after rejecting $\frac{cn}{m}$ as inconſiderable, we get $y^2 - bx - c = 0$. Conſequently the parabola expreſſed by this laſt equation is an aſymptote of the propoſed curve.

If both a and b be equal to o, the 2d aſympt. equat. entirely vaniſhes when $\frac{n}{m}$ is taken equal to o; and from the 1ſt aſympt. equat. we then have $y^2 - c = 0$, which is an equation to two right lines parallel to the baſe. From whence we have $y = c^{\frac{1}{2}}$, and $y = -c^{\frac{1}{2}}$. Therefore, in this caſe, the curve has two rectilinear aſymptotes parallel to the baſe, one above and the other below; and the diſtance of each from the baſe is $c^{\frac{1}{2}}$. And it is plain, theſe aſymptotes, when c is $= 0$, both coincide with the baſe.

It is obſervable, that, in every caſe, the principal ordinate is an aſymptote.

THE

THE

RESIDUAL ANALYSIS.

CHAP. X.

Of the DIAMETERS *and* CENTERS *of* curve Lines.

IN finding the forms and properties of curves, we frequently have occasion to enquire concerning their diameters and centers: It may therefore be worth while to ſhew the uſe of our doctrine in ſuch enquiries.

I.

When any number of parallel right lines, drawn between two branches of a curve, and terminated by thoſe branches, are all biſected by ſome line, I call ſuch biſecting line a *diameter* of the *firſt kind*. And, any number of parallel right lines being drawn between two extreme branches of a curve, and cutting the intermediate branches; if thoſe parallels be interſected by ſome line, (which is not ſuch a diameter as juſt now mentioned,) ſo that the aggregate of the parts of any parallel, lying on one ſide of that interſecting line, and terminated by it and a branch of the curve, be equal to the part, or aggregate of the parts, of the ſame parallel, terminated in the ſame manner, lying on the contrary ſide of the ſame interſecting line; I then, for diſtinction ſake, call ſuch interſecting line a *diameter* of the *ſecond kind*. When

all the rectilinear diameters of this second kind, appertaining to any curve, intersect each other in one and the same point; such point of intersection, I presume, may, not improperly, be called a *diametric pole*.

2.

A point bisecting all right lines drawn through it, and terminated by two branches of a curve, I call a *center* of the *first kind*. And, any number of right lines being drawn through a certain point, terminating at the extreme branches of a curve, and cutting the intermediate branches; if the aggregate of the parts of any such line, lying on one side of that point, and terminated by it and a branch of the curve, be equal to the part, or aggregate of the parts, of the same line, terminated in the same manner, lying on the contrary side of the same point; we may denominate such point a *center* of the *second kind*.

3.

Fig. 56. EFG being any curve, whose abscissa AB, is called x; and ordinate BF, y: suppose efg to be another line, whose abscissa Ab, and ordinate bf, (parallel to BF,) are denoted by v and w respectively: also suppose the right line Ff, which we will call u, always to make invariable angles with those ordinates. Then, drawing fh parallel to AB, and calling the sines of the angles fhF, fFh, and Ffh; k, m, and n respectively; fh will be $= \frac{mu}{k}$, and F$h = \frac{nu}{k}$. Consequently, writing z instead of $\frac{u}{k}$, x will be $= v + mz$, and $y = w + nz$. Which values of x and y being respectively substituted for their equals, in the equation of the curve EFG, the nature of that curve will be expressed by an equation shewing the relation of the three quantities v, w, and z; which may all vary together, whilst m and n remain invariable. And by expressing the nature of the curve in such manner, many useful conclusions relating thereto may be very easily inferred.

4. Suppose

4.

Suppose the equation of the curve EFG to be

$$ax^p y^q + bx^r y^s \text{ \&c.} = 0:$$

and suppose that, when $v + mz$ and $w + nz$ are therein respectively substituted for x and y, the resulting expression, viz.

$a.\overline{v+mz}|^p.\overline{w+nz}|^q + b.\overline{v+mz}|^r.\overline{w+nz}|^s$ &c. is $= A + Bz + Cz^2 + Dz^3 \ldots\ldots\ldots\ldots\ldots\ldots Pz^{p+q}$; $p+q$ denoting the order of the line EFG, and A, B, C, &c. being functions of v and w, in which z is not concerned. Then the value of $[t \mathbin{\underset{.}{-}} a.\overline{v+mz}|^p.\overline{w+nz}|^q] + [t \mathbin{\underset{.}{-}} b.\overline{v+mz}|^r.\overline{w+nz}|^s]$ &c. when only z is considered as variable, and unity is wrote instead of $[t \mathbin{\underset{.}{-}} z]$, being the same as when v and w are considered as variable, and z as invariable, and m and n are wrote instead of $[t \mathbin{\underset{.}{-}} v]$ and $[t \mathbin{\underset{.}{-}} w]$ respectively; A, B, C, &c. will be found as follows.

1st. From the above assumed equation, we, by taking z equal to 0, get $av^p w^q + bv^r w^s$ &c. $= A$.

2dly. By residual division, making $t - \underset{.}{t}$ the divisor, and, on one side of the assumed equation, considering v and w as variable whilst z is invariable, and writing m and n instead of $[t \mathbin{\underset{.}{-}} v]$ and $[t \mathbin{\underset{.}{-}} w]$ respectively; and, on the other side, considering z only as variable whilst A, B, C, &c. are invariable, and writing unity for $[t \mathbin{\underset{.}{-}} z]$; we get

$$amp.\overline{v+mz}|^{p-1}.\overline{w+nz}|^q + anq.\overline{v+mz}|^p.\overline{w+nz}|^{q-1}$$
$$+ bmr.\overline{v+mz}|^{r-1}.\overline{w+nz}|^s + bns.\overline{v+mz}|^r.\overline{w+nz}|^{s-1} \text{ \&c.}$$
$$= B + 2Cz + 3Dz^2 + 4Ez^3 \text{ \&c.}$$

Hence, by taking z equal to 0, we have

$$ampv^{p-1}w^q + anqv^p w^{q-1} + bmrv^{r-1}w^s + bnsv^r w^{s-1} \text{ \&c.} = B.$$

 And

And the values of C, D, E, &c. may be found in the same manner.

It is obvious therefore, that, if, in the equation of the curve EFG, (the terms being all brought to one side,) and the several asymptotic equations deduced therefrom by residual division, (as in the preceding chapter,) v and w be substituted instead of x and y respectively; the several expressions, after such substitution, will be the same as those by which (z^0, z, z^2, z^3, &c.) the several powers of z are respectively multiplied when $v + mz$ and $w + nz$ are substituted for x and y in the equation of the curve EFG, as before mentioned. And, by means of such expressions, we may not only determine the asymptotes of curves; but likewise, with the greatest facility, may find their diameters and centers; and moreover, discover many other remarkable particulars relative to the intersections of lines.

Example. *Let the curve be proposed whose equation is given in Examp. 2. of the last chapter: which equation is*

$$xy^2 + ey - ax^3 - bx^2 - cx - d = 0.$$

Then, transmuting that equation, as above mentioned, we have

$$\left.\begin{array}{r} vw^2 + ew - av^3 - bv^2 - cv - d \\ + \overline{mw^2 + 2nvw + en - 3amv^2 - 2bmv - cm} \times z \\ + \overline{2mnw + n^2v - 3am^2v - bm^2} \times z^2 \\ + \overline{mn^2 - am^3} \times z^3 \end{array}\right\} = 0.$$

Now, considering m and n as invariable whilst v, w, and z vary; if z has but two values, and one of them is always as much negative as the other is positive, the line efg, with respect to EFG, will be a diameter of the first kind: Which will be the case when the coefficients of z^3 and z vanish, and, at the same time, the coefficient of z^2 and the terms in which z is not concerned, do not vanish. Therefore, $\frac{m}{n}$ being as determined by the equation $mn^2 - am^3 = 0$, let the equation of the line efg be the coefficient of z put $= 0$, i. e.

$$mw^2 + 2nvw + en - 3amv^2 - 2bmv - cm = 0;$$

and

and that line will be a diameter of the firſt kind, with reſpect to the curve EFG. Moreover, $\frac{m}{n}$ having three real values when a is poſitive, there will then be three ſuch diameters; which will be expreſſed by the ſame equations, by which the hyperbolic aſymptotes of the curve EFG are expreſſed, in the ſecond Example in the preceding Chapter; and, efg being ſuch a diameter, Ff will be parallel to a rectilinear aſymptote of the ſame curve.

The equations expreſſing thoſe diameters are $2vw + e = 0$, $w^2 + 2a^{\frac{1}{2}}vw + a^{\frac{1}{2}}e - 3av^2 - 2bv - c = 0$, and $w^2 - 2a^{\frac{1}{2}}vw - a^{\frac{1}{2}}e - 3av^2 - 2bv - c = 0$. The firſt of which, when e is $= 0$, expreſſes two right lines, one coinciding with the baſe, and the other with the principal ordinate; whereof, the former is a diameter of the firſt kind, and the latter an aſymptote of the propoſed curve. By the ſecond of thoſe three equations, w is $= -a^{\frac{1}{2}}v \pm 2a^{\frac{1}{2}}\sqrt{v^2 + \frac{b}{2a}v + \frac{c - a^{\frac{1}{2}}e}{4a}}$; and, by the third, w is $= a^{\frac{1}{2}}v \pm 2a^{\frac{1}{2}}\sqrt{v^2 + \frac{b}{2a}v + \frac{c + a^{\frac{1}{2}}e}{4a}}$. It appears therefore, that if $\frac{b^2}{4a}$ be $= c - a^{\frac{1}{2}}e$, or $\frac{b^2}{4a} = c + a^{\frac{1}{2}}e$, the ſecond or third of thoſe three diametric equations will accordingly expreſs two right lines; one of which will be a diameter of the firſt kind, and the other an aſymptote of the curve EFG: and that, if e be $= 0$, and $\frac{b^2}{4a} = c$, the ſecond and third of the ſaid diametric equations will each expreſs two right lines; whereof, one will be a diameter, and the other an aſymptote, as juſt now mentioned; ſo that, in this laſt caſe, the propoſed curve will have three rectilinear diameters of the firſt kind.

If a be $= 0$, the ſecond and third diametric equations both become $w^2 - 2bv - c = 0$, an equation to a conical parabola; which differs a little from the parabola that, in the ſame caſe, is an aſymptote to the curve: and this diametric parabola will biſect lines drawn parallel to the baſe, and terminated by two branches of the propoſed curve.

If

If both a and b be equal to o, the coefficient of z^2 will vanifh when $\frac{n}{m}$ is $=$ o; which fhews that no right line parallel to the bafe can cut the curve EFG in more than one point. It is therefore obvious, that, in this cafe, (as well as when a is negative,) the curve will have but one diameter of the firft kind, which will be expreffed by the firft of the three diametric equations above written.

It is manifeft, that, if the equation of the line efg be the coefficient of z^2 put $=$ o, i. e. $2mnw + n^2v - 3am^2v - bm^2 = 0$; let $\frac{m}{n}$ be what it will, (provided it be not a root of the equation $\frac{m}{n} - \frac{am^3}{n^3} = 0$,) that line will be a diameter of the fecond kind, with refpect to the curve EFG.

And, $\frac{m}{n}$ being of any value whatever, when b is $=$ o, w in the laft mentioned diametric equation will always be equal to o when v is equal to o; and confequently the point where v (or x) begins will, in fuch cafe, be a diametric pole.

In finding a center of either kind, by our method, efg muft not be confidered as a line, but as a fixed point; and therefore v and w muft be confidered as invariable, in the coefficients of the feveral powers of z, whilft $\frac{m}{n}$ and z vary. Now, as $\frac{m}{n}$ muft be variable, it is plain, that $mn^2 - am^3$, the coefficient of z^3, cannot be equal to o. Therefore, for the propofed curve to have a center of the firft kind, the coefficient of z^2 muft vanifh without determining the value of $\frac{m}{n}$; and the terms wherein z is not concerned muft alfo vanifh, that the remaining terms in the equation of the curve may be divifible by z. Confequently, when a line of the third order has a center of the firft kind, fuch center will always be fome point in that line. Moreover, for ($2mnw + n^2v - 3am^2v - bm^2$) the coefficient of z^2 to vanifh without determining the value of

of $\frac{m}{n}$, it is evident b, v, and w muſt each be $= 0$; and ſince, the terms in which z is not concerned being ſuppoſed $= 0$, w cannot vaniſh when v vaniſhes, unleſs the term d be wanting; it follows, that the curve will have no center of the firſt kind, unleſs b and d be each $= 0$; and that, if the terms bx^2 and d be wanting in the propoſed equation, the point where the abſciſſa (x) begins will be a center of that kind.

It is obvious, that, if the coefficient of z^2 vaniſhes without determining the value of $\frac{m}{n}$, and, at the ſame time, the terms in which z is not concerned do not vaniſh; the curve will have a center of the ſecond kind. Which will be the caſe, when b, v, and w are each $= 0$; and the term d, at the ſame time, is not wanting: the point where the abſciſſa (x) begins being then ſuch a center.

This Chapter might be extended to a great length, in expatiating on the uſefulneſs of our method of tranſmuting the equation of a curve; but what is already ſaid may ſuffice to enable the intelligent Reader to make a farther application of it, at his pleaſure.

In the inveſtigation of propoſitions, as well phyſical as geometrical, the *relation between two variable quantities* is frequently the object of our enquiry; and ſuch relation never appears more clearly, than when one of thoſe quantities is conſidered as the *abſciſſa*, and the other as the *correſpondent ordinate* of a curve, and the form of the curve is properly aſcertained.—By conſidering the variable quantities in that light, miſtakes are prevented, and the moſt ſatisfactory concluſions obtained, in various intereſting diſquiſitions, particularly in the reſolution of problems relating to the *maxima* and *minima* of quantities.—Now, in aſcertaining the form of a curve from the equation thereof, the Articles in the laſt three Chapters will, I preſume, be found of conſiderable

considerable uſe; and therefore be acceptable to the ingenious lovers of Geometry: Who, finding the proceſſes eaſy and extenſive, will, I am perſuaded, eſteem their farther application a pleaſant exerciſe.

END *of the* FIRST BOOK.

Plate I.

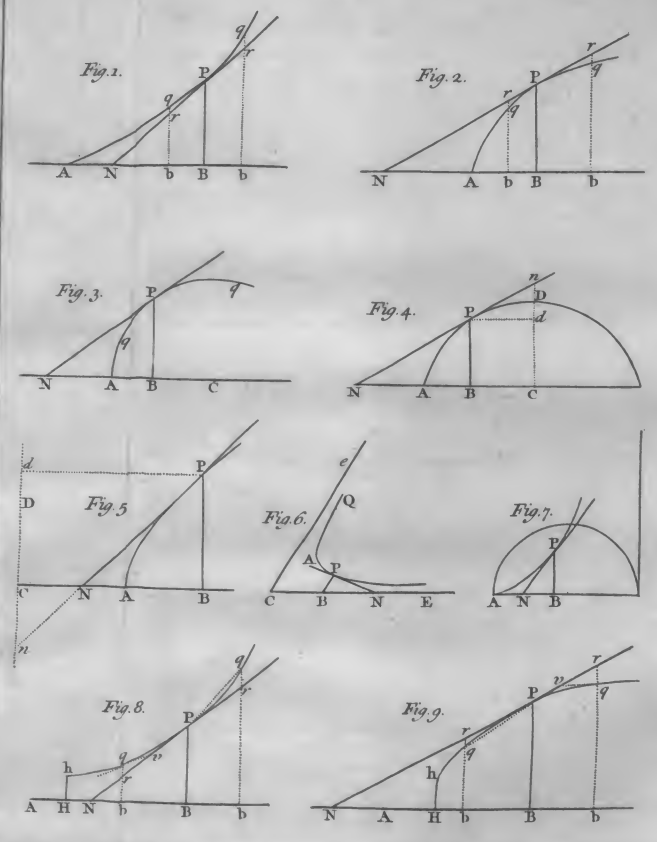

Plate II.

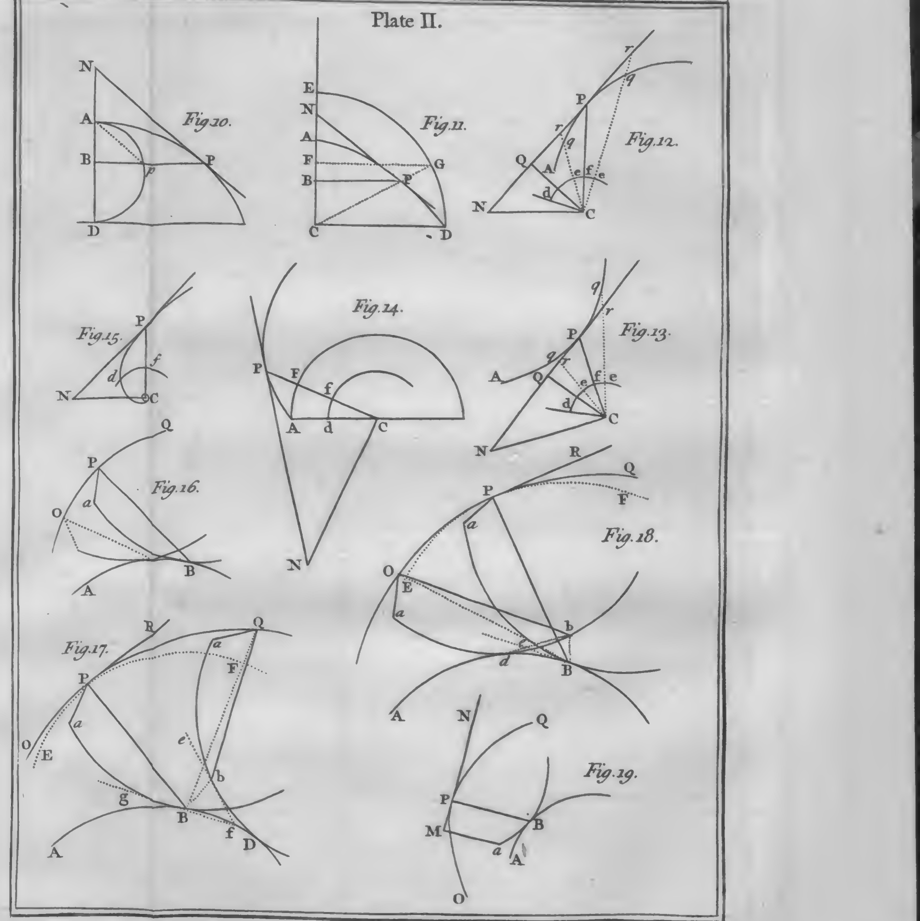

Fig.10.
Fig.11.
Fig.12.
Fig.13.
Fig.14.
Fig.15.
Fig.16.
Fig.17.
Fig.18.
Fig.19.

Plate III.

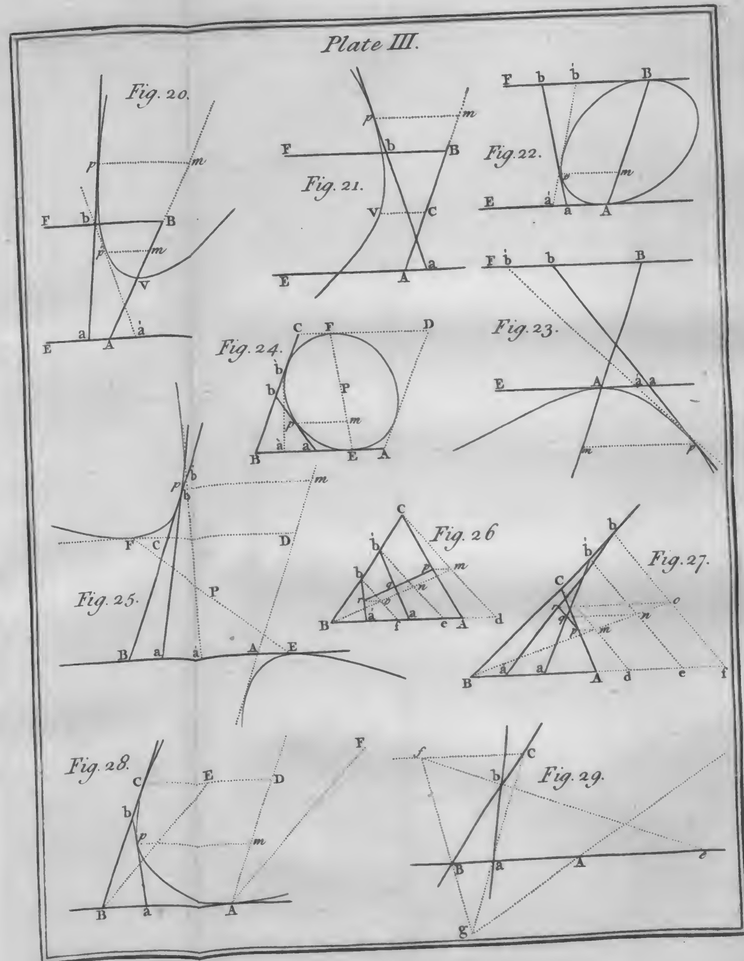
Fig. 20.
Fig. 21.
Fig. 22.
Fig. 23.
Fig. 24.
Fig. 25.
Fig. 26
Fig. 27.
Fig. 28.
Fig. 29.

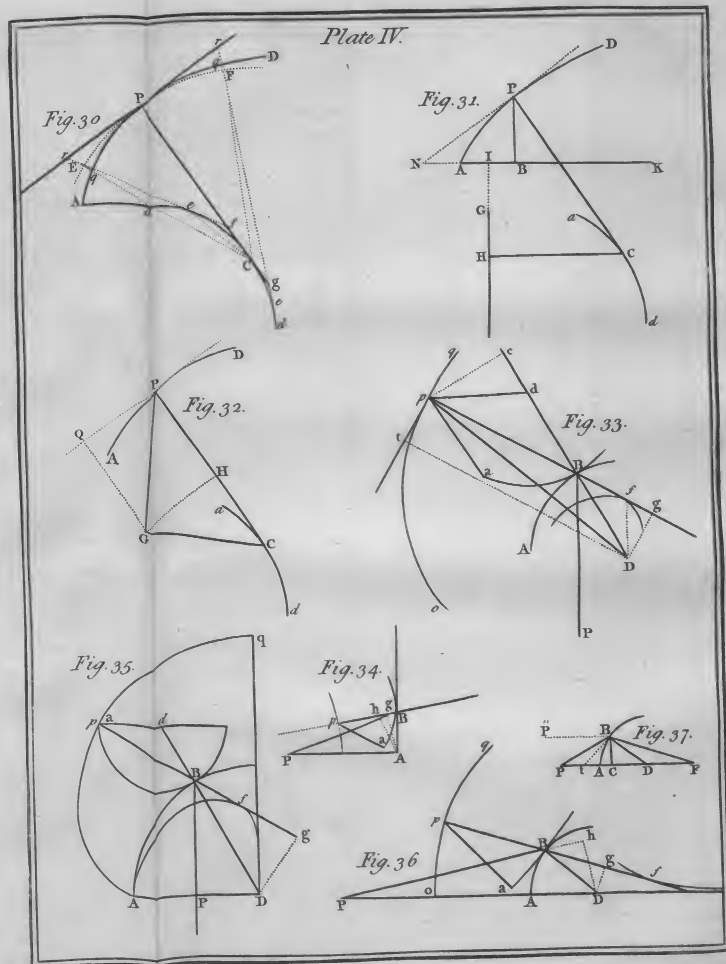
Plate IV.
Fig. 30.
Fig. 31.
Fig. 32.
Fig. 33.
Fig. 34.
Fig. 35.
Fig. 36.
Fig. 37.

Plate V.

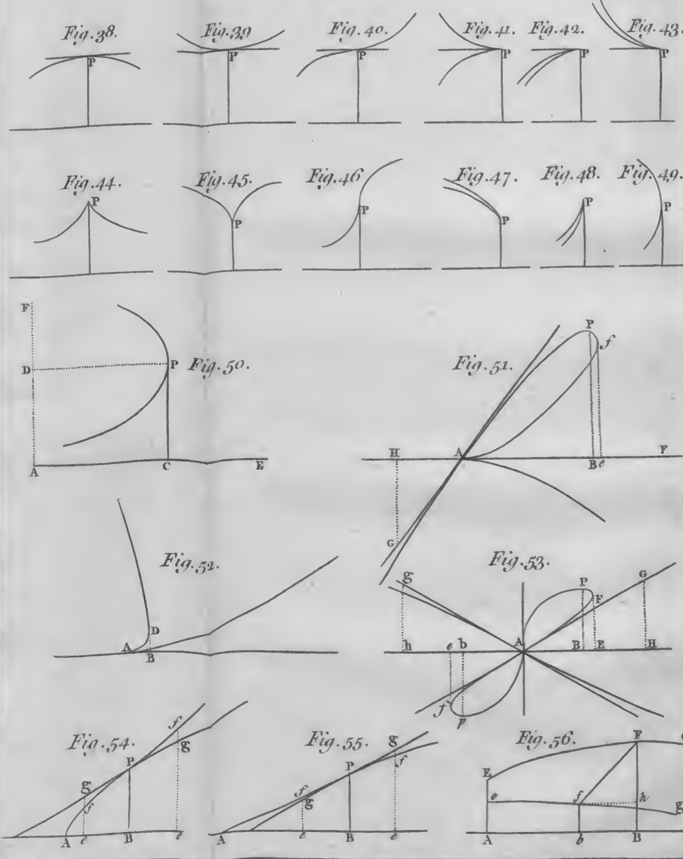

Fig. 38.
Fig. 39
Fig. 40.
Fig. 41.
Fig. 42.
Fig. 43.
Fig. 44.
Fig. 45.
Fig. 46
Fig. 47.
Fig. 48.
Fig. 49.
Fig. 50.
Fig. 51.
Fig. 52.
Fig. 53.
Fig. 54.
Fig. 55.
Fig. 56.